fiziks

Forum for CSIR-UGC JRF/NET, GATE, IIT-JAM/IISc, JEST, TIFR and GRE in PHYSICS & PHYSICAL SCIENCES

Electricity and Magnetism

(IIT-JAM/JEST/TIFR/M.Sc Entrance)

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ELECTRICITY AND MAGNETISM

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Questions & Solutions

Chapter - 1

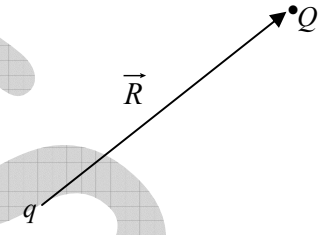
Electrostatics

The electric field at any point due to stationary source charges is called as electrostatic field.

1.1 Coulomb's Law and Superposition Principle

The electric force on a test charge Q due to a single point charge q, which is at rest and a distance R apart is given by Coulomb's law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{R^2}\hat{R}.$$



The constant ε_0 is called the *permittivity of free space*.

In mks units, $\varepsilon_0 = 8.85\times10^{-12}\,\frac{C^2}{N.m^2}$

$\vec{R}$ is the separation vector from $\vec{r'}$ (the location of q) to $\vec{r}$ (the location of Q): $\vec{R} = \vec{r} - \vec{r'}$; R is its magnitude, and $\hat{R}$ is its direction. The force points along the line from q to Q; it is repulsive if q and Q have the same sign, and attractive if their signs are opposite.

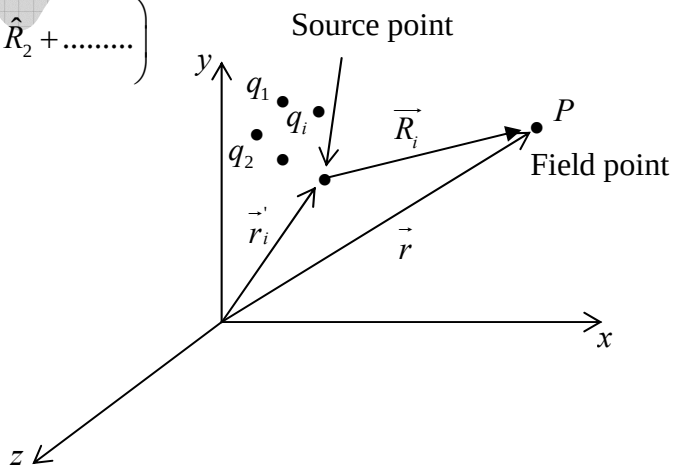
1.1.1 Electric Field

If we have many ***point charges*** $q_1, q_2, ...$ at distances $R_1, R_2, R_3.....$ from test charge Q, then according to the **principle of superposition** the total force on Q is

$$\vec{F} = \vec{F_1} + \vec{F_2} + ... = \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{R_1^2}\hat{R}_1 + \frac{q_2}{R_2^2}\hat{R}_2 + ...\right)$$

$$\Rightarrow \vec{F} = Q\vec{E}$$

where $\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i^2}\hat{R}_i$



$\vec{E}$ is called the **electric field** of the source charges. Physically $\vec{E}(P)$ is the force per unit charge that would be exerted on a test charge placed at P.

If ***charge is distributed continuously*** over some region, then

$$\vec{E}(r)=\frac{1}{4\pi\varepsilon_0}\int_{line}\frac{1}{R^2}\hat{R}dq \ .$$

The electric field of a line charge is $(dq=\lambda dl')$

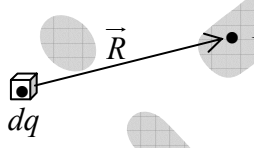
$$\vec{E}(r)=\frac{1}{4\pi\varepsilon_0}\int_{line}\frac{\lambda(r')}{R^2}\hat{R}dl' \text{ where } \lambda \text{ is charge per unit length.}$$

For surface charge $(dq=\sigma da')$

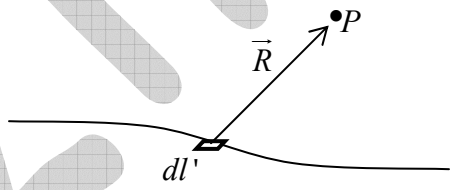
$$\vec{E}(r)=\frac{1}{4\pi\varepsilon_0}\int_{surface}\frac{\sigma(r')}{R^2}\hat{R}da' \text{ where } \sigma \text{ is charge per unit area.}$$

For a volume charge $(dq=\rho d\tau')$

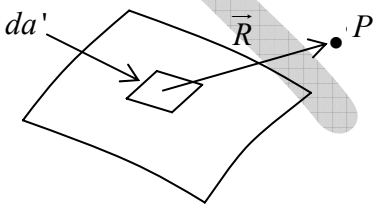
$$\vec{E}(r)=\frac{1}{4\pi\varepsilon_0}\int_{volume}\frac{\rho(r')}{R^2}\hat{R}d\tau' \text{ where } \rho \text{ is charge per unit volume.}$$



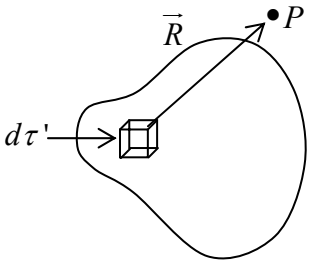
(a) Continuous distribution



(b) Line charge, λ



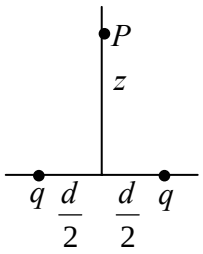
(c) Surface charge, σ



(d) Volume charge, ρ

Example: (a) Find the Electric a distance z above the mid point between two equal charges, q, a distance d apart.

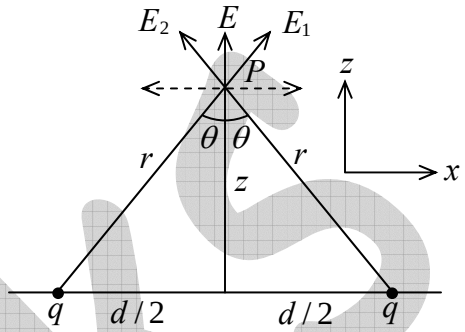
(b) Repeat part (a) after replacing right hand charge to $-q$.



Solution: (a) $E_1 = E_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$

Horizontal components cancels and

$$E_z = E_1\cos\theta + E_2\cos\theta = 2E_1\cos\theta$$

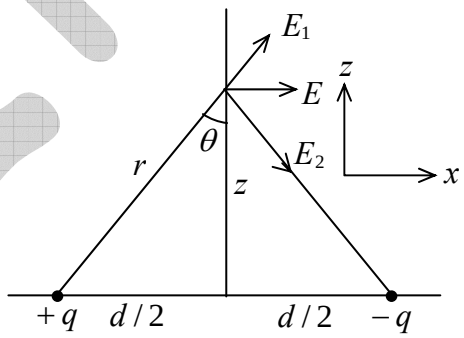


Since $r^2 = z^2 + \dfrac{d^2}{4}, \cos\theta = \dfrac{z}{r}$

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2qz}{\left[z^2+\frac{d^2}{4}\right]^{3/2}}\hat{z}$$

When $z >> d$, $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{2q}{z^2}$ (looks like a single charge $2q$).

(b) $E_1 = E_2 = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}$



Component along z-direction cancel out.

Thus $\vec{E} = 2E_1\sin\theta\,\hat{x}$, $\sin\theta = \dfrac{d}{2r}$

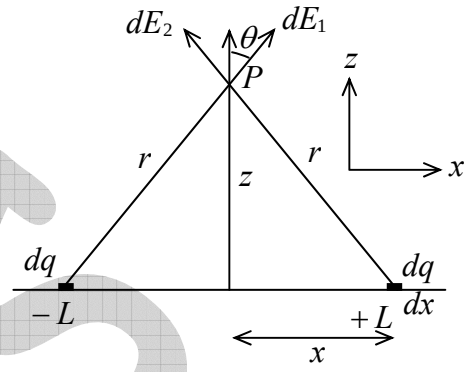
$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qd}{\left[z^2+\frac{d^2}{4}\right]^{3/2}}\hat{x}$$

When $z >> d$, $\vec{E} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{qd}{z^3}\hat{x}$ (field of a dipole)

Example: Find the electric field a distance z above the midpoint of a straight line segment of length $2L$, which carries a uniform line charge λ.

Solution: Horizontal components of two field cancels and the field of the two segment is

$$dE_1 = dE_2 = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}$$



Net field is $d\vec{E} = 2dE_1\cos\theta\hat{z} = 2\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda dx}{r^2}\right)\cos\theta\hat{z}$

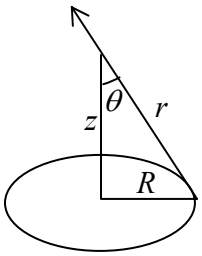
Here, $\cos\theta = \frac{z}{r}, r = \sqrt{z^2+x^2} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{2\lambda z}{\left[z^2+x^2\right]^{3/2}}dx$

Thus $E = \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]_0^L \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}\hat{z}$

For $z >> L$, $\mathrm{E} \approx \frac{1}{4\pi\varepsilon_0}\frac{2\lambda \mathrm{L}}{\mathrm{z}^2}$ and when $L \to \infty$, $E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$

Example: Find the electric field a distance z above the center of a circular loop of radius r, which carries a uniform line charge λ.

Solution: "Horizontal" components cancel, leaving:



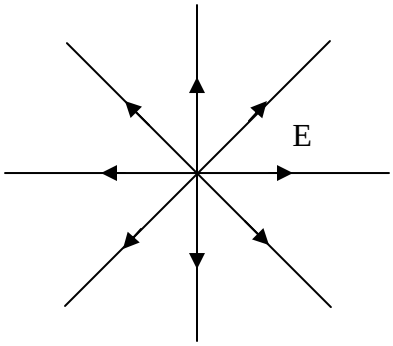
$$\vec{\mathrm{E}} = \frac{1}{4\pi\varepsilon_0}\left\{\int\frac{\lambda \mathrm{d}l}{\mathrm{r}^2}\cos\theta\right\}\hat{\mathrm{z}}$$

Here $r^2 = R^2 + z^2$, $\cos\theta = \frac{z}{r}$ (both constants),

$$\Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda(2\pi R)z}{(R^2+z^2)^{3/2}}\hat{z} \quad \because \int dl = 2\pi R$$

1.2 Gauss's Law

1.2.1 Field Lines and Electric Flux

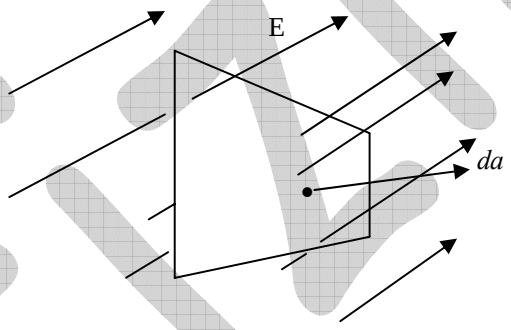


Consider that a point charge q is situated at the origin:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

This field is represented by the **field line** as shown in figure below. The magnitude of the field is indicated by the density of the field lines: it's strong near the center where the field line are close together, and weak farther out, where they are relatively far apart.

The **field strength** (E) is proportional to the number of field lines per unit area (area perpendicular to the lines). The **flux of** E through a surface *S*, $\phi_E = \int_S \vec{E}.d\vec{a}$ is a measure of the "number of field lines" passing through *S*.



For the case of point charge at the origin, the flux of $\vec{E}$ through a sphere of radius r is

$$\oint \vec{E}.d\vec{a} = \int \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r^2}\hat{r}\right).\left(r^2\sin\theta d\theta d\phi\,\hat{r}\right) = \frac{1}{\varepsilon_0}q\,.$$

Note that, any surface whatever its shape, would trap the same number of field lines. So the flux through any surface enclosing the charge is $\frac{q}{\varepsilon_0}$.

Now suppose that instead of a single charge at the origin, we have a bunch of charges scattered about. According to the principle of superposition, the total field is simply the (vector) sum of all the individual fields: $\vec{E} = \sum_{i=1}^{n}\vec{E}_i$.

The flux through any surface that encloses them all, then, is

$$\oint \vec{E}.d\vec{a} = \sum_i^n \left(\oint \vec{E}_i .d\vec{a} \right) = \sum_i^n \left(\frac{1}{\varepsilon_0} q_i \right).$$

A charge outside the surface would contribute nothing to the total flux, since its field lines go in one side and out the other. It follows, then, that for any closed surface,

$$\oint \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

where Q_{enc} is the total charge enclosed within the surface. This is **Gauss's law in integral form.**

We can convert Gauss's law in **integral form** to **differential form**, for continuous charge distributions, by applying the divergence theorem:

$$\oint_{surface} \vec{E}.d\vec{a} = \int_{volume} \left(\vec{\nabla}.\vec{E} \right) d\tau .$$

We can write Q_{enc} in terms of the charge density ρ, we have $Q_{enc} = \int_{volume} \rho d\tau$.

So Gauss's law becomes $\int_{volume} \left(\vec{\nabla}.\vec{E} \right) d\tau = \int_{volume} \left(\frac{1}{\varepsilon_0} \rho \right) d\tau .$

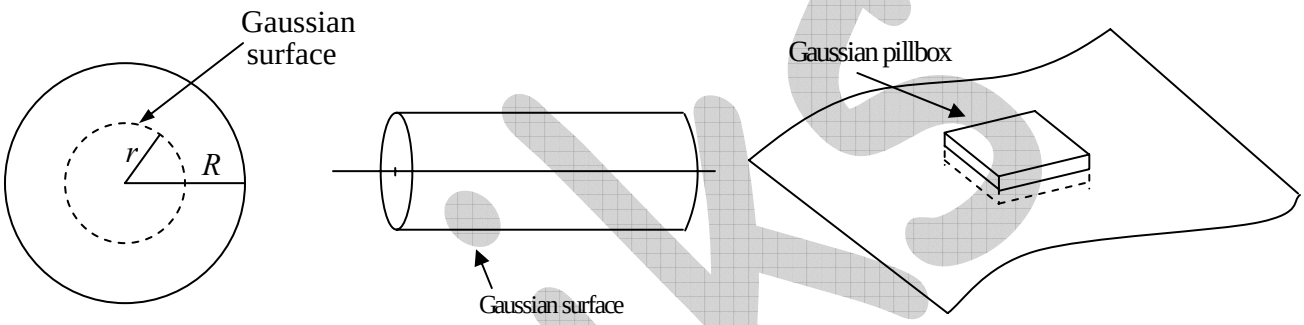
Since this holds for any volume, the integrands must be equal:

$$\vec{\nabla}.\vec{E} = \frac{1}{\varepsilon_0} \rho .$$

1.2.2 Applications of Gauss's Law

Gauss's law is always true, but it is not always useful. Gauss's law is useful for only three kinds of symmetry:

1. **Spherical Symmetry**. Make your Gaussian surface a concentric sphere.
2. **Cylindrical Symmetry**. Make your Gaussian surface a coaxial cylinder.
3. **Plane Symmetry**. Make your Gaussian surface a "pillbox," which extends equally above and below the surface.



Example: Find the field inside and outside a uniformly charged solid sphere of radius R and total charge q.

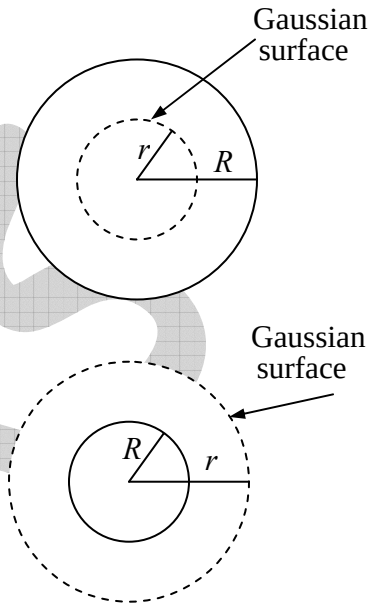
Solution: Draw a spherical surface of radius $r < R$, which is called as "Gaussian surface".

According Gauss's Law

$$\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc} \Rightarrow \int_S |\vec{E}|da = |\vec{E}|\int_S da = |\vec{E}|\times 4\pi r^2 = \frac{1}{\varepsilon_0}Q_{enc}$$

$$|\vec{E}|\times 4\pi r^2 = \frac{1}{\varepsilon_0}\left\{\frac{q}{\frac{4}{3}\pi R^3}\times\frac{4}{3}\pi r^3\right\}$$

$$\Rightarrow |\vec{E}| = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3} \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r}$$

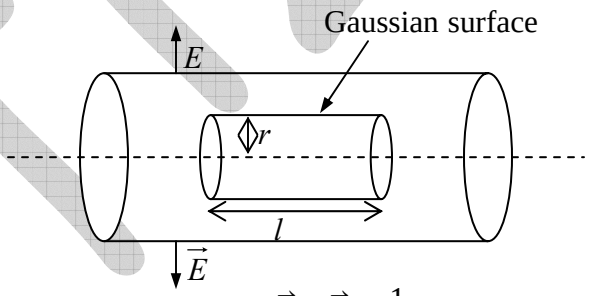


For outside point, draw a spherical surface of radius $r > R$,

$$|\vec{E}|\times 4\pi r^2 = \frac{1}{\varepsilon_0}q \Rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Example: A long cylinder carries a charge density that is proportional to the distance from the axis; $\rho = kr$, for some constant k. Find the electric field inside this cylinder.

Solution: Draw a Gaussian cylinder of length l and radius r.



For this surface, Gauss's Law states: $\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0}Q_{enc}$

Now, $Q_{enc} = \int \rho d\tau = \int (kr')(r'dr'd\phi dz) = 2\pi kl\int_0^r r'^2 dr' = \frac{2\pi}{3}klr^3$.

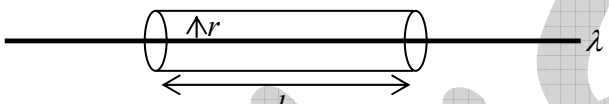
Symmetry dictates that $\vec{E}$ must point radially outward and the two ends contribute nothing to electric flux as $\vec{E} \perp d\vec{a}$.

For curved portion $\int \vec{E}.d\vec{a} = \int |\vec{E}|\,da = |E| \int da = |E| \times 2\pi rl = \frac{1}{\varepsilon_0}\frac{2\pi}{3} klr^3$

$$\Rightarrow \vec{E} = \frac{1}{3\varepsilon_0} kr^2 \hat{r}\,.$$

Example: Find the electric field a distance *r* from an infinitely long straight wire, which carries a uniform line charge λ.

Solution: Draw a Gaussian cylinder of length l and radius r.

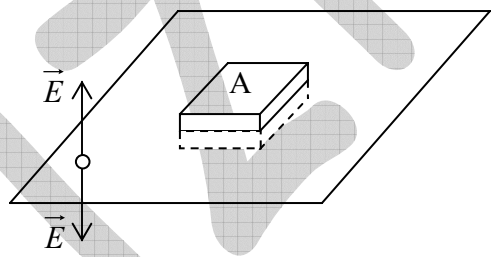


For this surface, Gauss's Law state: $\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc} \Rightarrow E \times 2\pi rl = \frac{1}{\varepsilon_0}\lambda l$

$$\Rightarrow \vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Example: An infinite plane carries a uniform surface charge σ. Find its electric field.

Solution: Draw a "Gaussian pill box", extending equal distances above and below the plane.



Apply Gauss's Law to this surface: $\oint_S \vec{E}.d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$

In this case, $Q_{enc} = \sigma A$, where A is the area of the pill box. By symmetry, $\vec{E}$ points away from the plane (upward for the points above, downward for points below).

Thus $\int \vec{E}.d\vec{a} = 2A \times |\vec{E}|$ whereas sides contribute nothing. Thus $2A \times |\vec{E}| = \frac{1}{\varepsilon_0}\sigma A$

$$\Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} \text{ where } \hat{n} \text{ is the unit vector pointing away from the surface}$$

Example: Suppose the electric field in some region is found to be $\vec{E} = Ar^3\hat{r}$, in spherical coordinates (A is some constant).

(a) Find the charge density ρ.

(b) Find the total charge contained in a sphere of radius R, centered at the origin.

Solution:

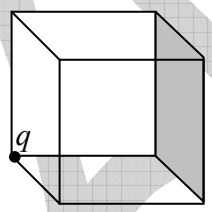
(a) $\rho = \varepsilon_0 \vec{\nabla}.\vec{E} = \varepsilon_0 \left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 E_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r^2\sin\theta}\frac{\partial E_\phi}{\partial\phi} \right]$

$\because E_r = Ar^3,\ E_\theta = 0,\ E_\phi = 0 \Rightarrow \rho = \varepsilon_0 \frac{1}{r^2} A(5r^4) = 5\varepsilon_0 Ar^2$

(b) By Gauss's Law, $Q_{enc} = \varepsilon_0 \oint \vec{E}.d\vec{a} = \varepsilon_0 (AR^3)(4\pi R^2) = 4\pi\varepsilon_0 AR^5$

Also, $Q_{enc} = \int \rho d\tau = \int_0^R (5\varepsilon_0 Ar^3)(4\pi r^2 dr) = 20\pi\varepsilon_0 A\int_0^R r^4 dr = 4\pi\varepsilon_0 AR^5$

Example: A charge q sits at the back corner of a cube as shown in figure. What is the flux of $\vec{E}$ through the shaded side?



Solution: Think of this cube as one of 8 surrounding the charge. Each of the 24 squares which make up the surface of this larger cube gets the same flux as every other one.

$$\int_{\text{one face}} \vec{E}.d\vec{a} = \frac{1}{24}\int_{\substack{\text{whole}\\ \text{large}\\ \text{cube}}} \vec{E}.d\vec{a} = \frac{q}{24\varepsilon_0}$$

1.3 Electric Potential

1.3.1 Curl of Electric field

Consider a point charge at the origin, then electric field at a distance r is given by

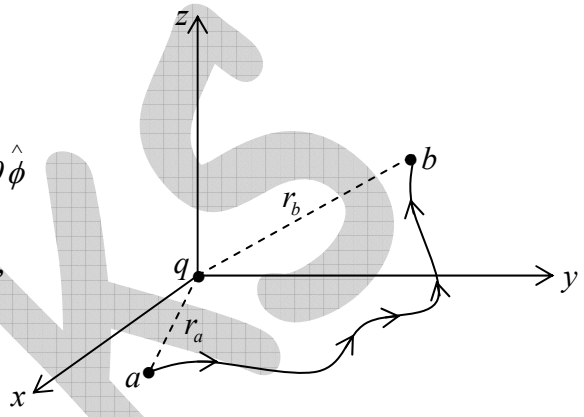
$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.$$

Now we will calculate the line integral of this field from some point a to some other point

$$b: \int_a^b \vec{E}.d\vec{l}\ .$$

In spherical coordinates, $d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\hat{\phi}$

$$\Rightarrow \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr \Rightarrow \int_a^b \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_a^b \frac{q}{r^2}dr,$$



$$\Rightarrow \int_a^b \vec{E}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}q\left(\frac{1}{r_a} - \frac{1}{r_b}\right)$$ where r_a is the distance from the origin to point a and r_b is the distance to b.

Then integral around a closed path is zero i.e. $\oint \vec{E}.d\vec{l} = 0 \ \ (\because r_a = r_b)$

This line integral is independent of path. It depends on two end points.

Applying stokes theorem, we get $\vec{\nabla}\times\vec{E} = 0$. The electric field is not just any vector but only those vector whose curl is zero.

If we have many charges, the principle of superposition states that the total field is the vector sum of their individual fields:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + ...$$

So, $$\vec{\nabla}\times\vec{E} = \vec{\nabla}\times\vec{E}_1 + \vec{\nabla}\times\vec{E}_2 + ... = 0$$

Since $\oint \vec{E}.d\vec{l} = 0$, the line integral is independent of path.

So, we can define a function $V(r) = -\int_{\vartheta}^{P} \vec{E}.d\vec{l}$

where ϑ is some standard reference point V then depends only on the point r. It is called the **electric potential.**

Evidently, the potential difference between two points a and b is

$$V(b)-V(a)=-\int_{\vartheta}^{b}\vec{E}.d\vec{l}+\int_{\vartheta}^{a}\vec{E}.d\vec{l},$$

$$V(b)-V(a)=-\int_{\vartheta}^{b}\vec{E}.d\vec{l}-\int_{a}^{\vartheta}\vec{E}.d\vec{l}\Rightarrow V(b)-V(a)=-\int_{a}^{b}\vec{E}.d\vec{l}.$$

Now, the fundamental theorem for gradients states that

$$V(b)-V(a)=\int_{a}^{b}\left(\vec{\nabla}.V\right).d\vec{l}=-\int_{a}^{b}\vec{E}.d\vec{l}$$

$$\Rightarrow \vec{E}=-\vec{\nabla}V.$$

Potential obeys the superposition principle.

1.3.2 Potential of localized charges

Potential of a point charge q is $V=\frac{1}{4\pi\varepsilon_0}\frac{q}{R}$ where R is the distance from the charge.

The potential of a collection of point charge is $V=\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i}$.

For continuous volume charge distribution $V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(r')}{R}d\tau'$

The potential of line and surface charges are $V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\lambda(r')}{R}dl'$ and $V(r)=\frac{1}{4\pi\varepsilon_0}\int\frac{\sigma(r')}{R}da'$.

Example: Which one of these is an impossible electrostatic field?

(a) $\vec{E} = k[xy\hat{x} + 2yz\hat{y} + 3xz\hat{z}]$

(b) $\vec{E} = k[y^2\hat{x} + (2xy + z^2)\hat{y} + 2yz\hat{z}]$

Solution: (a) $\vec{\nabla} \times \vec{E}_1 = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = k\left[\hat{x}(0-2y) + \hat{y}(0-3z) + \hat{z}(0-x)\right] \neq 0$

So, $\vec{E}_1$ is an impossible electrostatic field.

(b) $\vec{\nabla} \times \vec{E}_2 = 0$ so $\vec{E}_2$ is a possible electrostatic field.

Example: Find the potential inside and outside a ***spherical shell*** of radius R, charge q.

Solution: From Gauss's law the field

$$\vec{E} = \begin{cases} \vec{E}_1 = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}; & \text{outside the sphere } (r > R) \\ \vec{E}_2 = 0; & \text{inside the sphere } (r < R) \end{cases}$$

Potential outside $(r > R)$ is: $V(r) = -\int_\infty^r \vec{E}_1.d\vec{l} = -\int_\infty^r \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}.dr\hat{r} = \dfrac{q}{4\pi\varepsilon_0 r}$

Potential inside $(r < R)$ is: $V(r) = -\int_\infty^r \vec{E}.d\vec{l} = -\int_\infty^R \vec{E}_1.d\vec{l} - \int_R^r \vec{E}_2.d\vec{l} = \dfrac{q}{4\pi\varepsilon_0 R}$

So potential inside the spherical shell is constant.

Thus $V(r) = \dfrac{q}{4\pi\varepsilon_0 R}$; $r \le R$ and $V(r) = \dfrac{q}{4\pi\varepsilon_0 r}$; $r > R$.

Example: Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q.

Solution: From Gauss's law the field

$$\vec{E} = \begin{cases} \vec{E}_1 = \dfrac{q}{4\pi\varepsilon_0 r^2}\hat{r}; & \text{outside the sphere } (r > R) \\ \vec{E}_2 = \dfrac{qr}{4\pi\varepsilon_0 R^3}\hat{r}; & \text{inside the sphere } (r < R) \end{cases}$$

Potential outside $(r > R)$ is: $V(r) = -\int_{\infty}^{r} \vec{E}_1 . d\vec{l} = -\int_{\infty}^{r} \frac{q}{4\pi\varepsilon_0 r^2} dr = \frac{q}{4\pi\varepsilon_0 r}$

Potential inside $(r < R)$ is: $V(r) = -\int_{\infty}^{r} \vec{E} . d\vec{l} = -\int_{\infty}^{R} \vec{E}_1 . d\vec{l} - \int_{R}^{r} \vec{E}_2 . d\vec{l}$

$$\Rightarrow V(r) = -\int_{\infty}^{R} \left(\frac{1}{4\pi\varepsilon_0} \frac{q}{r'^2} \right) dr' - \int_{R}^{r} \left(\frac{1}{4\pi\varepsilon_0} \frac{qr'}{R^3} \right) dr' = \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{R} - \frac{1}{R^3} \left(\frac{r^2 - R^2}{2} \right) \right]$$

$$\Rightarrow V(r) = \frac{q}{4\pi\varepsilon_0} \frac{1}{2R} \left(3 - \frac{r^2}{R^2} \right)$$

Example: Find the potential a distance r from an infinitely long straight wire that carries a uniform line charge λ.

Solution: since $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r} \hat{r}$. In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at $r = a$

Then $V(r) = -\int_{a}^{r} \left(\frac{1}{2\pi\varepsilon_0} \frac{\lambda}{r'} \right) dr' = -\frac{\lambda}{2\pi\varepsilon_0} \ln\left(\frac{r}{a} \right)$

1.4 Laplace's and Poisson Equations

Since $\vec{E} = -\vec{\nabla} V$ and $\vec{\nabla} . \vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\varepsilon_0}$

This is known as **Poisson's equation**.

In regions where there is no charge, so that $\rho = 0$, Poisson's equation reduces to Laplace's equation, $\nabla^2 V = 0$.

Example: Consider two concentric spherical conducting shells centered at the origin. The outer radius of the inner shell is r_a and the inner radius of the outer shell is r_b. The charge ensity $\rho = 0$ in the region $r_a < r < r_b$. If $V = 0 \; at \; r = r_a$ and $V = V_0 \; at \; r = r_b$, then find V in the region $r_a < r < r_b$.

Solution: Since voltage is varying only with *r*, the Laplace's equation takes the form

$$\nabla^2 V = \frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

Integrate twice to get the solution $V(r) = A\ln(r) + B$

and the boundary conditions are

(i) $V = 0 \; at \; r = r_a$ (ii) $V = V_0 \; at \; r = r_b$

Substituting these boundary conditions, we get

At $r = r_a$, $0 = A\ln r_a + B \Rightarrow B = -A\ln r_a$

At $r = r_b$, $V_0 = A\ln r_b + B \Rightarrow V_0 = A\ln r_b - A\ln r_a = A\ln\left(\frac{r_b}{r_a}\right)$

$$\Rightarrow A = \frac{V_0}{\ln\left(\frac{r_b}{r_a}\right)} \text{ and } B = -\frac{V_0}{\ln\left(\frac{r_b}{r_a}\right)}\ln r_a$$

Thus $V = \frac{V_0}{\ln\left(\frac{r_b}{r_a}\right)}\ln r - \frac{V_0 \ln r_a}{\ln\left(\frac{r_b}{r_a}\right)} = \frac{V_0}{\ln\left(\frac{r_b}{r_a}\right)}(\ln r - \ln r_a) \Rightarrow V = \frac{V_0 \ln\left(\frac{r}{r_a}\right)}{\ln\left(\frac{r_b}{r_a}\right)}$

Example: Potential in a region of space is given by, $\phi = \phi_0 e^{-ax^2}$ where ϕ_0 and a is constant. Then find the charge density in this region.

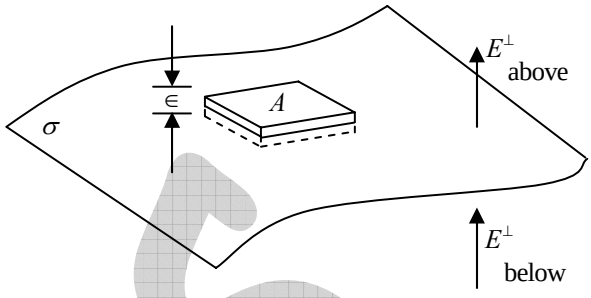
Solution: $\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \Rightarrow \rho = -\varepsilon_0(\nabla^2\phi) = 2a\varepsilon_0\phi(1 - 2ax^2)$

Example: If the electrostatic potential were given by $\phi = \phi_0(x^2 + y^2 + z^2)$, where ϕ_0 is constant, then find the charge density giving rise to the above potential.

Solution: $\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \Rightarrow \rho = -6\phi_0\varepsilon_0$

1.5 Electrostatic Boundary Condition

The boundary between two medium is a thin sheet of surface charge σ. Consider a thin Gaussian pillbox, extending equally above and below the sheet as shown in figure below:



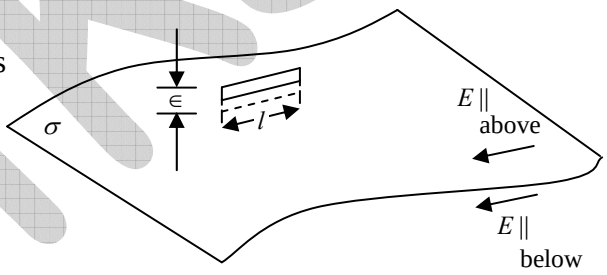
The Gauss's law states that $\oint_S \vec{E}.d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$.

$$\Rightarrow E^{\perp}_{above}A - E^{\perp}_{below}A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\varepsilon_0}.$$

The normal component of $\vec{E}$ is discontinuous by an amount $\frac{\sigma}{\varepsilon_0}$ at any boundary. If there is no surface charge, $E^{\perp}$ is continuous.

The tangential component of $\vec{E}$ is always continuous.



Apply $\oint \vec{E}.d\vec{l} = 0$ to the thin rectangular loop,

$$E^{\|}_{above}l - E^{\|}_{below}l = 0 \Rightarrow E^{\|}_{above} = E^{\|}_{below}$$

where $\vec{E}^{\|}$ stands for the components of $\vec{E}$ parallel to the surface.

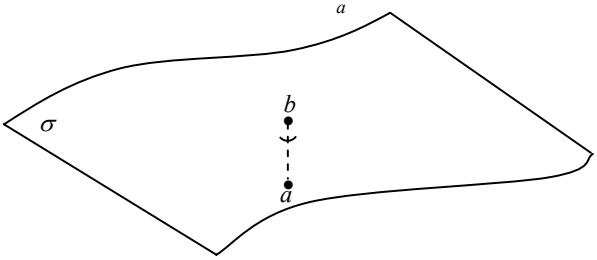
The boundary conditions on $\vec{E}$ can be combined into single formula:

$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

where $\hat{n}$ is unit vector perpendicular to the surface, pointing upward.

The potential is continuous across any boundary, since $V_{above} - V_{below} = -\int_a^b \vec{E}.d\vec{l}$; as the path shrinks to zero.

$$\Rightarrow V_{above} = V_{below}.$$

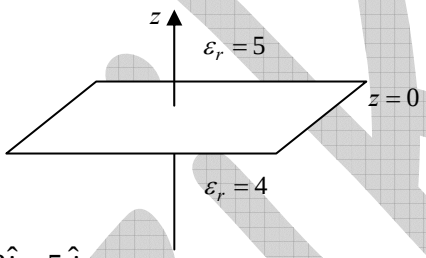


Since $\vec{E} = -\vec{\nabla}V \Rightarrow \vec{\nabla}V_{above} - \vec{\nabla}V_{below} = -\frac{\sigma}{\varepsilon_0}\hat{n}$,

$$\Rightarrow \frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

where $\frac{\partial V}{\partial n} = \vec{\nabla} V \cdot \hat{n}$ denotes the normal derivative of V (that is the rate of change in the direction perpendicular to the surface.)

Example: Assume that $z = 0$ plane is the interface between two linear and homogenous dielectrics (see figure). The relative permittivities are $\varepsilon_r = 5$ for $z > 0$ and $\varepsilon_r = 4$ for $z < 0$. The electric field in the region $z > 0$ is $\vec{E}_1 = \left(3\hat{i} - 5\hat{j} + 4\hat{k}\right) k \ \ V/m$. If there are no free charges on the interface, then find electric field in the region $z < 0$.



Solution: $\because E_1^{II} = E_2^{II} \Rightarrow E_2^{II} = 3\hat{i} - 5\hat{j}$

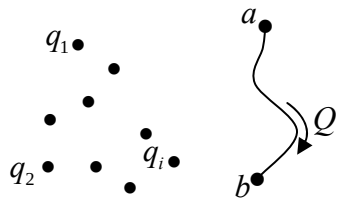
and $\sigma_f = 0 \ \Rightarrow D_1^{\perp} = D_2^{\perp} \Rightarrow E_2^{\perp} = \frac{\varepsilon_1}{\varepsilon_2} E_1^{\perp} = \frac{5}{4}\left(+4\hat{k}\right) = 5\hat{k}$

$\Rightarrow \vec{E}_2 = \left(3\hat{i} - 5\hat{j} + 5\hat{k}\right) k\, V/m$

1.6 Work and Energy in Electrostatics

The work done in moving a test charge Q in an external field $\vec{E}$, from point a to b is

$$W = \int_a^b \vec{F} \cdot d\vec{l} = -Q \int_a^b \vec{E} \cdot d\vec{l} = Q\left[V(b) - V(a)\right]$$



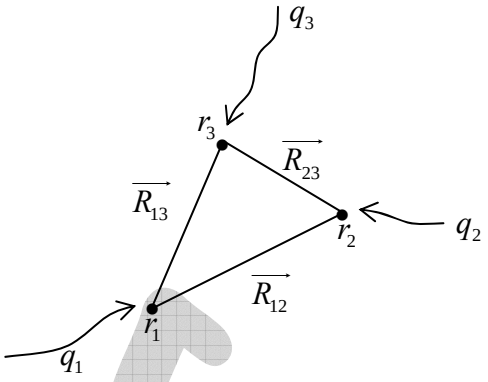
If $a = \infty$ and $b = r$

$$\Rightarrow W = Q\left[V(r) - V(\infty)\right] = QV(r) \ \text{ since } V(\infty) = 0$$

In this sense **potential** is potential energy (the work it takes to create the system) per unit charge (just as the field is the force per unit charge).

1.6.1 The Energy of Point Charge Distribution

When the first charge q_1 is placed, no work has been done. When q_2 is placed work done $W_2 = q_2 V_1$ where V_1 is the potential due to q_1 so, $W_2 = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{R_{12}} \right)$.

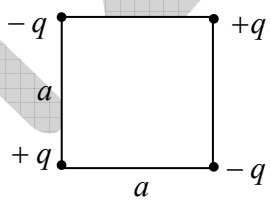


Similarly when third charge q_3 is placed $W_3 = \frac{1}{4\pi\varepsilon_0} q_3 \left(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}} \right)$

The work necessary to assemble the first three charges is $W = \frac{1}{4\pi\varepsilon_0} \left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} \right)$

In general, $W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{R_{ij}} = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{q_i q_j}{R_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i V(r_i)$, where $V(r_i)$ is the potential at point r_i (the position of q_i) due to all other charges.

Example: Four charges are situated at the corners of a square (side a) as shown in figure. How much work does it take to assemble the whole configuration of four charges?



Solution: Work done in placing first charge ($-q$ charge upper left corner) $W_1 = 0$

Work done in placing second charge ($+q$ charge lower left corner) $W_2 = \frac{1}{4\pi\varepsilon_0} \left(-\frac{q^2}{a} \right)$

Work done in placing third charge ($-q$ charge lower right corner)

$$W_3 = \frac{1}{4\pi\varepsilon_0} \left(-\frac{q^2}{a} + \frac{q^2}{\sqrt{2}a} \right)$$

Potential at fourth corner ($+q$ charge upper right corner)

$$V=\frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{r_i}=\frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{a}+\frac{q}{\sqrt{2}a}-\frac{q}{a}\right)=\frac{q}{4\pi\varepsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow W_4=qV=\frac{q^2}{4\pi\varepsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

Total work done $=W_1+W_2+W_3+W_4=\frac{1}{4\pi\varepsilon_0}\frac{2q^2}{a}\left(-2+\frac{1}{\sqrt{2}}\right)=\frac{q^2}{2\pi\varepsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$

1.6.2 Energy of Continuous Charge Distribution

For a volume charge density $W=\frac{1}{2}\int\rho V d\tau$,

Since $\rho=\varepsilon_0\left(\vec{\nabla}.\vec{E}\right)\Rightarrow W=\frac{\varepsilon_0}{2}\int\left(\vec{\nabla}.\vec{E}\right)Vd\tau\Rightarrow W=\frac{\varepsilon_0}{2}\left[-\int_V\vec{E}.\left(\vec{\nabla}V\right)d\tau+\int_V\vec{\nabla}.\left(V\vec{E}\right)d\tau\right]$

$$\Rightarrow W=\frac{\varepsilon_0}{2}\left[\int_V E^2 d\tau+\oint_S V\vec{E}.d\vec{a}\right]\ \text{Since}\ \vec{E}=-\vec{\nabla}V$$

The above equation gives the correct energy W, whatever volume we use as long as it encloses all the charges, but the contribution from the volume integral goes up, and that of the surface integral goes down, as we take larger and larger volumes. In particular, if we integrate over all space, then the surface integral goes to zero, and we have

$$W=\frac{\varepsilon_0}{2}\int_{all\ space}E^2 d\tau$$

Example: Energy of a uniformly charged spherical shell of total charge q and radius R.

Solution: $\vec{E}_{inside}=0,\qquad \vec{E}_{outside}=\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$

$$W=\frac{\varepsilon_0}{2}\int_{all\ space}E^2 d\tau=\frac{\varepsilon_0}{2}\int_0^R E_{in}^2 d\tau+\frac{\varepsilon_0}{2}\int_R^\infty E_{out}^2 d\tau=\frac{\varepsilon_0}{2(4\pi\varepsilon_0)^2}\int_{outside}\left(\frac{q^2}{r^4}\right)(r^2\sin\theta dr d\theta d\phi)$$

$$W=\frac{1}{32\pi^2\varepsilon_0}q^2 4\pi\int_R^\infty\frac{1}{r^2}dr\quad\Rightarrow W=\frac{q^2}{8\pi\varepsilon_0 R}.$$

Example: Find the energy stored in a uniformly charged solid sphere of radius R and charge q.

Solution: $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r};\ r > R$ *and* $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r};\ r < R$

$$W = \frac{\varepsilon_0}{2}\frac{q^2}{(4\pi\varepsilon_0)^2}\left\{\int_R^\infty \frac{1}{r^4}(r^2 4\pi dr) + \int_0^R \left(\frac{r}{R^3}\right)^2 (4\pi r^2 dr)\right\} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{R} + \frac{1}{5R}\right\},$$

$$W = \frac{1}{4\pi\varepsilon_0}\frac{3q^2}{5R}$$

1.7 Basic Properties of Conductors

1. $\vec{E} = 0$ inside a conductor.
2. $\rho = 0$ inside a conductor.
3. Any net charge resides on the surface.
4. A conductor is an equipotential.
5. $\vec{E}$ is perpendicular to the surface, just outside a conductor.

Because the field inside a conductor is zero, boundary condition $\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$ requires that the field immediately outside is

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{n}.$$

In terms of potential equation $\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} = -\frac{\sigma}{\varepsilon_0}$ yields

$$\sigma = -\varepsilon_0\frac{\partial V}{\partial n}.$$

These equations enable us to calculate the surface charge on a conductor, if we can determine $\vec{E}$ or V.

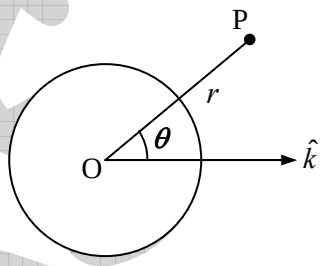
Force per unit area on the conductor is $\vec{f} = \frac{1}{2\varepsilon_0}\sigma^2\hat{n}$.

This amounts to an outwards *electrostatic pressure* on the surface, tending to draw the conductor into the field, regardless the sign of σ. Expressing the pressure in terms of the field just outside the surface,

$$P = \frac{\varepsilon_0}{2} E^2 .$$

Example: A spherical conductor of radius a is placed in a uniform electric field $\vec{E} = E_0 \hat{k}$. The potential at a point $P(r,\theta)$ for $r > a$, is given by

$$\Phi(r,\theta) = constant - E_0 r \sin\theta + \frac{E_0 a^3}{r^2}\sin\theta$$

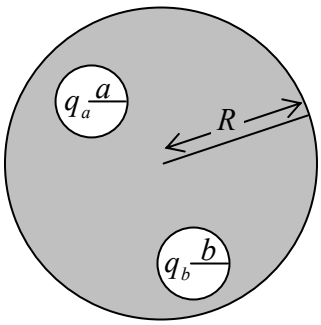


where r is the distance of P from the centre O of the sphere and θ is the angle OP makes with the z-axis. Then the charge density on the sphere at $\theta = 60^0$ is

Solution: $\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial r}\right|_{r=a} = -\varepsilon_0 \left[-E_0 \sin\theta - \frac{2E_0 a^3}{r^3}\sin\theta \right]_{r=a}$

$$\sigma = -\varepsilon_0 \left[-E_0 \sin\theta - 2E_0 \sin\theta \right] \Rightarrow \sigma = +3E_0\varepsilon_0 \sin\theta = +3E_0\varepsilon_0 \sin 60^0 = \frac{3\sqrt{3}}{2}\varepsilon_0 E_0$$

Example: Two spherical cavities, of radii a and b, are hollowed out from the interior of a conducting sphere of radius R. At the center of each cavity a point charge is placed, call these charges q_a and q_b.



(a) Find the surface charges σ_a, σ_b and σ_R.

(b) What is the field outside the conductor?

(c) What is the field within each cavity?

(d) What is the force on q_a and q_b?

(e) If a third charge q_c, were brought near the conductor, which of the above answers will change.

Solution: (a) $\sigma_a = \frac{-q_a}{4\pi a^2}$, $\sigma_b = \frac{-q_b}{4\pi b^2}$, $\sigma_R = \frac{q_a + q_b}{4\pi R^2}$.

(b) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_a + q_b}{r^2}\hat{r};\ r > R$

(c) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_a}{r^2}\hat{r};\ r < a \quad and \quad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_b}{r^2}\hat{r};\ r < b.$

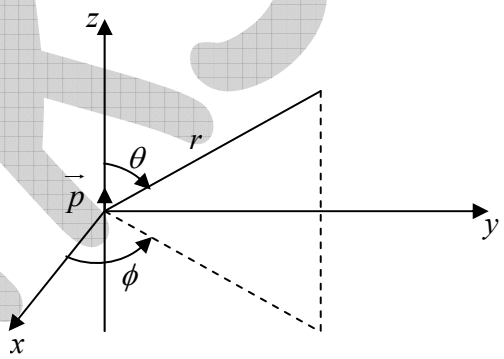
(d) Zero.

(e) σ_R will change and hence electric field outside the conductor will also change.

1.8 Multipole Expansions

1.8.1 The Electric Potential and Field of a Dipole

If we choose coordinates so that $\vec{p}$ (dipole moment) lies at the origin and points in the z direction, then potential at (r,θ) is:



$$V_{dip}(r,\theta) = \frac{\hat{r}.\vec{p}}{4\pi\varepsilon_o r^2} = \frac{p\cos\theta}{4\pi\varepsilon_o r^2}.$$

Since $\vec{E} = -\vec{\nabla}V$

$$\Rightarrow E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_o r^3}, E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{p\sin\theta}{4\pi\varepsilon_o r^3}, E_\phi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi} = 0.$$

The electric field of a dipole:

$$\vec{E}_{dip}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\hat{\theta}\right).$$

We can express $\vec{p} = \left(\vec{p}.\hat{r}\right)\hat{r} + \left(\vec{p}.\hat{\theta}\right)\hat{\theta} = p\cos\theta\hat{r} - p\sin\theta\hat{\theta}$.

Thus $3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p} = 2p\cos\theta\,\hat{r} + p\sin\theta\hat{\theta}$

$$\Rightarrow \vec{E}_{dip}(r,\theta) = \frac{1}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p}\right]$$

Note:

(a) When a dipole is placed in a uniform electric field ($\vec{E}$), net force on the dipole is zero and it experiences a torque $\vec{\tau} = \vec{p} \times \vec{E}$ where $\vec{p} = q\vec{d}$.

(b) In non-uniform field, dipoles have net force $\vec{F} = \left(\vec{p} \cdot \vec{\nabla}\right)\vec{E}$ and torque $\vec{\tau} = \vec{p} \times \vec{E}$.

(c) Energy of an ideal dipole $\vec{p}$ in an electric field $\vec{E}$ is $U = -\vec{p}.\vec{E}$.

(d) Interaction energy of two dipoles separated by a distance $\vec{r}$ is

$$U = \frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{p_1} \cdot \vec{p_2} - 3\left(\vec{p_1} \cdot \hat{r}\right)\left(\vec{p_2} \cdot \hat{r}\right)\right]$$

Example: A "pure" dipole $\vec{p}$ is situated at the origin, pointing in the *z*-direction

(a) What is the force on a point charge q at $(a,0,0)$?

(b) What is the force on q at $(0,0,a)$?

(c) How much work does it take to move q from $(a,0,0)$ to $(0,0,a)$?

Solution: (a) $At\ (a,0,0),\ r = a, \theta = \frac{\pi}{2};\ \vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\hat{\theta}\right) = \frac{p}{4\pi\varepsilon_0 a^3}\hat{\theta}$

$$\vec{E} = -\frac{p}{4\pi\varepsilon_0 a^3}\hat{z} \Rightarrow \vec{F} = q\vec{E} = -\frac{pq}{4\pi\varepsilon_0 a^3}\hat{z}$$

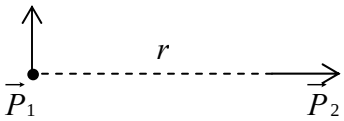
(b) $At\ (0,0,a),\ r = a, \theta = 0;\ \vec{E} = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\hat{\theta}\right) = \frac{2p}{4\pi\varepsilon_0 a^3}\hat{r}$

$$\vec{E} = \frac{2p}{4\pi\varepsilon_0 a^3}\hat{z} \Rightarrow \vec{F} = q\vec{E} = \frac{2pq}{4\pi\varepsilon_0 a^3}\hat{z}$$

(c) $V_{dip}(r,\theta) = \frac{p\cos\theta}{4\pi\varepsilon_o r^2} \Rightarrow V\ (a,0,0) = 0 \ \ and \ \ V\ (0,0,a) = \frac{p}{4\pi\varepsilon_o a^2}$,

$$W = q\left[V\ (0,0,a) - V\ (a,0,0)\right] = \frac{p}{4\pi\varepsilon_o a^2}$$

Example: In figure $\vec{p}_1$ and $\vec{p}_2$ are (perfect) dipoles a distance *r* apart. What is the torque on $\vec{p}_1$ due to $\vec{p}_2$? What is the torque on $\vec{p}_2$ due to $\vec{p}_1$?



Solution: $\vec{\tau}_1 = \vec{p}_1 \times \vec{E}_2$, where $\vec{E}_2 = \frac{p_2}{4\pi\varepsilon_0 r^3}\left(2\cos\pi\,\hat{r} + \sin\pi\hat{\theta}\right) = \frac{-2p_2}{4\pi\varepsilon_0 a^3}\hat{r}$,

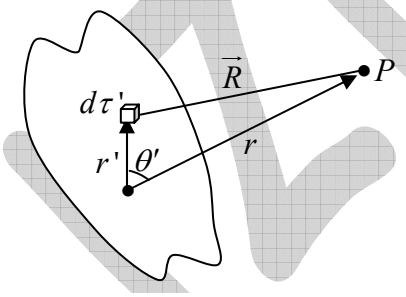
$$\tau_1 = \frac{2p_1p_2}{4\pi\varepsilon_0 a^3} \quad \text{(inward)}$$

$\vec{\tau}_2 = \vec{p}_2 \times \vec{E}_1$, where $\vec{E}_1 = \frac{p_1}{4\pi\varepsilon_0 r^3}\left(2\cos\frac{\pi}{2}\hat{r} + \sin\frac{\pi}{2}\hat{\theta}\right) = \frac{p_2}{4\pi\varepsilon_0 a^3}\hat{\theta}$,

$$\tau_2 = \frac{p_1p_2}{4\pi\varepsilon_0 a^3} \quad \text{(inward)}$$

1.8.2 Approximate potential at large distances

Approximate potential at large distances due to arbitrary localized charge distribution



$$V(r) = \frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r}\int\rho(r')\,d\tau' + \frac{1}{r^2}\int r'\cos\theta'\rho(r')\,d\tau' + \frac{1}{r^2}\int(r')^2\left(\frac{3}{2}\cos^2\theta' - \frac{1}{2}\right)\rho(r')\,d\tau' + \ldots\right]$$

The first term $(n=0)$ is the monopole contribution (it goes like $\frac{1}{r}$). The second term $(n=1)$ is the dipole term (it goes like $\frac{1}{r^2}$). The third term is quadrupole; the fourth octopole and so on.

The lowest nonzero term in the expansion provides the approximate potential at large *r* and the successive terms tell us how to improve the approximation if greater precision is required.

The Monopole and Dipole Terms

Ordinarily, the multipole expansion is dominated (at large r) by the monopole term:

$$V_{mon}(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

where $Q = \int \rho d\tau$ is the total charge of the configuration.

If the total charge is zero, the dominant term in the potential will be the dipole (unless, of course, it also vanishes):

$$V_{dip}(r) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r' \cos\theta' \rho(r') d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}.\int \vec{r}' \rho(r') d\tau' = \frac{1}{4\pi\varepsilon_0}\frac{\hat{r}.\vec{p}}{r^2},$$

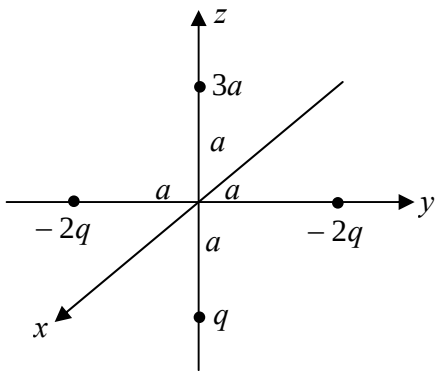
where ***dipole moment*** $\vec{p} = \int \vec{r}' \rho(r') d\tau'$

The dipole moment is determined by the geometry (size, shape and density) of the charge distribute. The dipole moment of a collection of point charge is

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i'$$

Note: Ordinarily, the dipole moment does change when we shift the origin, but there is an important exception: If the total charge is zero, then the dipole moment is independent of the choice of origin.

Example: Find the approximate potential for points on the z–axis, far from the sphere. Four particles of charges q, $3q$, $-2q$ and $-2q$ are placed as shown in figure, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin.



Solution: Net dipole moment $\vec{p} = (3qa - qa)\hat{z} + (-2qa - 2q \times -a)\hat{y} = 2qa\hat{z}$

Therefore $V \cong \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\hat{r}}{r^2}$ and $\vec{p}.\hat{r} = 2qa\hat{z}.\hat{r} = 2qa\cos\theta \Rightarrow V \cong \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2}$

Example: An insulating sphere of radius R carries a charge density

$$\rho\left(\vec{r}\right)=\rho\left(R^2-r^2\right)\cos^2\theta;\quad r<R$$

Find the leading order term for the electric field at a distance d, far away from the charge distribution.

Solution: $Q=\int\rho d\tau=\rho\int\left(R^2-r^2\right)\cos^2\theta\times r^2\sin\theta dr\,d\theta\,d\phi\neq 0\ \Rightarrow V\propto\frac{1}{d}\Rightarrow E\propto\frac{1}{d^2}$

Example: A sphere of radius R, centered at the origin, carries charge density $\rho(r,\theta)=k\frac{R}{r^2}(R-2r)\sin\theta$, where k is a constant, and r, θ are the usual spherical coordinates.

Solution: **<u>Monopole term</u>:** $Q=\int\rho d\tau=kR\int\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta dr\,d\theta\,d\phi=0$

Since the r integral is $\int_0^R(R-2r)dr=0$.

<u>Dipole term</u>:

$\vec{p}=\int\vec{r}\,\rho\left(r'\right)d\tau'=\int r\cos\theta\,\rho d\tau=KR\int(r\cos\theta)\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta dr\,d\theta\,d\phi=0$,

Since the integral is $\int_0^\pi\sin^2\theta\cos\theta d\theta=\left.\frac{\sin^3\theta}{3}\right|_0^\pi=0$.

<u>Quadrupole term</u>:

$$\int r^2\left(\frac{3}{2}\cos^2\theta-\frac{1}{2}\right)\rho d\tau=\frac{1}{2}kR\int r^2(3\cos^2\theta-1)\left[\frac{1}{r^2}(R-2r)\sin\theta\right]r^2\sin\theta\,dr\,d\theta$$

r integral : $\int_0^R r^2(R-2r)dr=\left.\left(\frac{r^3}{3}R-\frac{r^4}{2}\right)\right|_0^R=\frac{R^4}{3}-\frac{R^4}{2}=\frac{-R^4}{6}$

θ integral: $\int_0^\pi(3\cos^2\theta-1)\sin^2\theta d\theta=2\int_0^\pi\sin^2\theta d\theta-3\int_0^\pi\sin^4\theta d\theta=2\times\frac{\pi}{2}-3\times\frac{3\pi}{8}=-\frac{\pi}{8}$.

ϕ integral: $\int_0^{2\pi}d\phi=2\pi$.

$\Rightarrow$whole integral $=\frac{1}{2}kR\times\left(\frac{-R^4}{6}\right)\times\left(\frac{-\pi}{8}\right)\times 2\pi=\frac{k\pi^2R^5}{48}$.

For point P on the z–axis $\left(r\to z\right)$ the approximate potential is $V(z)\cong\frac{1}{4\pi\in_0}\frac{k\pi^2R^5}{48z^3}$.

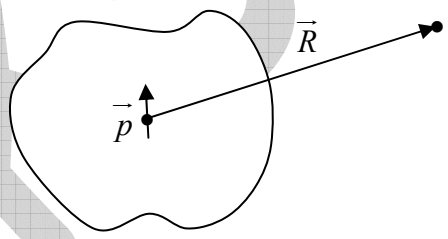
1.9 Polarization

When a piece of dielectric material is placed in an external field and if the substance consists of neutral atoms, the field will induce in a tiny dipole moment, pointing in the same direction as the field. If the material is made up of polar molecules each permanent dipole will experience a torque, tending to line it up along the field direction. (Random thermal motions compete with this process, so the alignment is never complete, especially at higher temperatures, and disappears almost at once when the field is removed.)

(Polarization) $\vec{P} \equiv$ dipole moment per unit volume

1.9.1 The Field of a Polarized Object (Bound Charges)

Suppose we have a piece of polarized material with polarization vector $\vec{P}$ containing a lot of microscopic dipoles lined up.



For a single dipole of dipole moment $\vec{p}$ we have $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}.\hat{R}}{R^2}$ where $\vec{R}$ is the vector from the dipole to the point at which we are evaluating the potential.

Thus $V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\hat{R}.\vec{P}(r')}{R^2} d\tau'$; since $\vec{p} = \vec{P}(r')d\tau'$

By solving the above equation, we get $V\left(\vec{r}\right) = \frac{1}{4\pi\varepsilon_0}\oint_S \frac{1}{R}\vec{P}.d\vec{a}' - \frac{1}{4\pi\varepsilon_0}\int_V \left(\frac{1}{R}\right)\left(\vec{\nabla}'.\vec{P}\right)d\tau'$

The first term looks like the potential of a ***surface bound charge*** $\sigma_b = \vec{P}.\hat{n}$ (where $\hat{n}$ is the normal unit vector)

The second term looks like the potential of a ***volume bound charge*** $\rho_b = -\vec{\nabla}.\vec{P}$

Thus potential (and hence also the field) of a polarized object is the same as that produced by a volume charge density $\rho_b = -\vec{\nabla}.\vec{P}$ plus a surface charge density $\sigma_b = \vec{P}.\hat{n}$.

Example: A sphere of radius R carries a polarization $\vec{P}(r) = K\vec{r}$ where K is a constant and $\vec{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.

(b) Find the field inside and outside the sphere.

Solution:

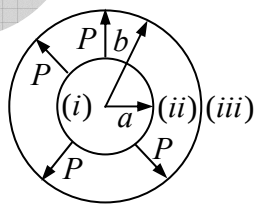
(a) $\sigma_b = \vec{P}.\hat{n} = kR\hat{r}.\hat{r} = kR; \quad \rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 kr) = -3k$

(b) For $r < R$; $\vec{E} = \frac{\rho r}{3\varepsilon_0}\hat{r} \Rightarrow \vec{E} = \frac{-kr}{\varepsilon_0}\hat{r}$

For $r > R$; same as if all charge at center; but

$Q_{total} = (kR)\times 4\pi R^2 + (-3k)\times\frac{3}{4}\pi R^3 = 0 \Rightarrow \vec{E} = 0$

Example: A thick spherical shell (inner radius a and outer radius b) is made of dielectric material with a polarization $\vec{P}(r) = \frac{k}{r}\hat{r}$, where k is a constant and r is the distance from the center. Find the electric field in all three regions.



Solution: $\rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = \frac{-k}{r^2}$ and $\sigma_b = \vec{P}.\hat{n} = \begin{cases} +\vec{P}.\hat{r} = \frac{k}{b} & (at\ r = b) \\ -\vec{P}.\hat{r} = \frac{-k}{a} & (at\ r = a) \end{cases}$

For $r < a$; $Q_{enc} = 0 \Rightarrow \vec{E} = 0$

For $r > b$; $Q_{enc} = \oint_S \sigma\, da + \oint_v \rho_b d\tau = \oint_s \vec{P}.d\vec{a} - \oint_v \left(\nabla.\vec{P}\right)d\tau = 0 \quad \left(\because \oint_s \vec{P}.d\vec{a} = \oint_v \vec{\nabla}.\vec{P}\, d\tau\right)$

$\Rightarrow \vec{E} = 0$

For $a < r < b$; $Q_{enc} = \left(\frac{-k}{a}\right)\times 4\pi a^2 + \int_a^r \left(\frac{-k}{\bar{r}^2}\right)4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k\left(r - a\right) = -4\pi kr$

$\because \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q_{enc}}{r^2} \Rightarrow \vec{E} = \frac{-k}{\varepsilon_0 r}\hat{r}$

1.10 The Electric Displacement

1.10.1 Gauss Law in the Presence of Dielectrics

Within the dielectric, the total charge density can be written as $\rho = \rho_b + \rho_f$ where ρ_b is volume bound charge ρ_f **free charge** density.

From Gauss Law; $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \varepsilon_0\left(\vec{\nabla}.\vec{E}\right) = \left(\rho_b + \rho_f\right) = -\vec{\nabla}.\vec{P} + \rho_f$ where $\vec{E}$ is now the **total field**, not just that portion generated by polarization.

$\Rightarrow \vec{\nabla}.\left(\varepsilon_0\vec{E} + \vec{P}\right) = \rho_f \Rightarrow \vec{\nabla}.\vec{D} = \rho_f$ where $\vec{D} = \varepsilon_0\vec{E} + \vec{P}$ is known as the **electric displacement.**

Thus Gauss' law reads, $\vec{\nabla}.\vec{D} = \rho_f$

or, in integral form $\oint \vec{D}.d\vec{a} = Q_{f_{enc}}$, where $Q_{f_{enc}}$ denotes the total free charge enclosed in the volume.

1.10.2 Linear Dielectrics (Susceptibility, Permittivity, Dielectric Constant)

For any substances, the polarization is proportional to the field provided $\vec{E}$ is not too strong: $\vec{P} \propto \vec{E} \Rightarrow \vec{P} = \varepsilon_0\chi_e\vec{E}$

(Materials that obey this relation are called linear dielectrics)

The constant of proportionality, χ_e is called the ***electric susceptibility*** of the medium. The value of χ_e depends on the microscopic structure of the substance and also on external conditions such as temperature.

In linear media we have

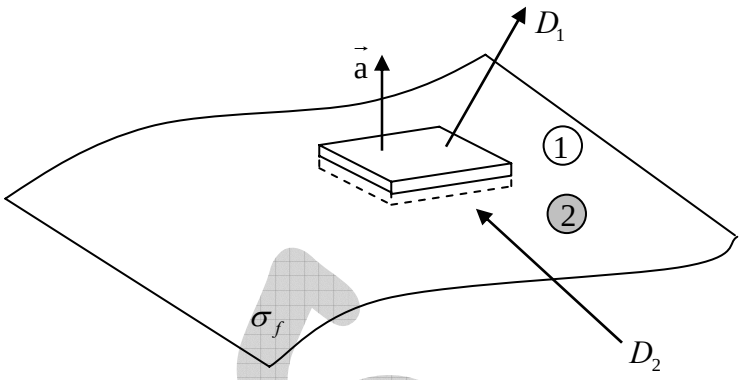
$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} = \varepsilon_0\vec{E} + \varepsilon_0\chi_e\vec{E} = \varepsilon_0\vec{E}\,(1+\chi_e) = \varepsilon\vec{E}, \quad where \quad \varepsilon = \varepsilon_0(1+\chi_e)$$

This new constant ε is called the permittivity of the material.

Also $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0} = (1+\chi_e)$ is called ***relative permittivity*** or ***dielectric constant***, of the material.

1.10.3 Boundary Condition on $\vec{D}$

The boundary between two medium is a thin sheet of free surface charge σ_f.



The Gauss's law states that

$$\oint_S \vec{D} d\vec{a} = Q_{free} \Rightarrow D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f$$

Since $\vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{\nabla} \times \vec{D} = \vec{\nabla} \times \vec{P}$

$$\Rightarrow \vec{D}^{\parallel}_{above} - \vec{D}^{\parallel}_{below} = \vec{P}^{\parallel}_{above} - \vec{P}^{\parallel}_{below}$$

$\left(\because \vec{\nabla} \times \vec{E} = 0\right)$

1.10.4 Energy in dielectric system

$$W = \frac{1}{2} \int_{all\ space} \left(\vec{D} \cdot \vec{E}\right) d\tau .$$

Example: A metal sphere of radius a carries a charge Q. It is surrounded, out to radius b, by linear dielectric material of permittivity ε. Find the potential at the center.

Solution: $\oint \vec{D}.d\vec{a} = Q_{f_{enc}} \Rightarrow \quad \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$; for all points $r > a$

(Inside the metal sphere, $\vec{E} = \vec{P} = \vec{D} = 0$). Once we know $\vec{D}$, it is a trivial matter to obtain $\vec{E}$ by $\vec{D} = \varepsilon \vec{E}$).

$$\vec{E} = \begin{cases} \frac{Q}{4\pi\varepsilon r^2}\hat{r} & \text{for } a < r < b \\ \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & \text{for } r > b \\ 0 & \text{for } r < a. \end{cases}$$

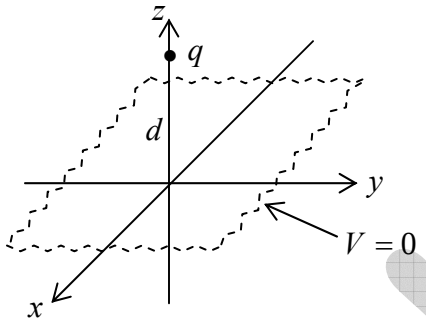
Potential at the center is therefore

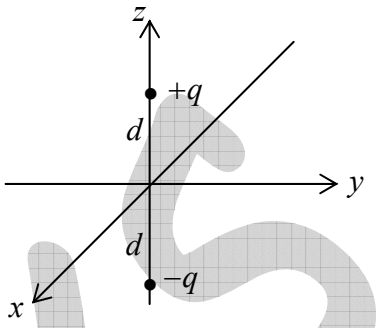
$$V = -\int_{\infty}^{0} \vec{E}.d\vec{l} = -\int_{\infty}^{b}\left(\frac{Q}{4\pi\varepsilon_0 r^2}\right)dr - \int_{b}^{a}\left(\frac{Q}{4\pi\varepsilon r^2}\right)dr - \int_{a}^{0}(0)dr \Rightarrow V = \frac{Q}{4\pi}\left(\frac{1}{\varepsilon_0 b} + \frac{1}{\varepsilon a} - \frac{1}{\varepsilon b}\right)$$

1.11 Image Problems

1.11.1 The Classic Image Problem

Suppose a point charge q is held a distance d above an infinite grounded conducting plane. We can find out what is the potential in the region above the plane.





Forget about the actual problem; we are going to study a *complete different situation*.

The new problem consists of two point charges $+q$ at $(0,0,d)$ and $-q$ at $(0,0,-d)$ and no conducting plane. For this configuration we can easily write down the potential:

$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}-\frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

(The denominators represent the distances from (x,y,z) to the charges $+q$ and $-q$, respectively.) It follows that

1. $V=0$ when $z=0$ and

2. $V\to 0$ for $x^2+y^2+z^2>>d^2$,

and the only charge in the region $z>0$ is the point charge $+q$ at $(0,0,d)$. Thus the second configuration produces exactly the same potential as the first configuration, in the upper region $z\geq 0$.

1.11.2 Induced Surface Charge

The surface charge density σ induced on the conductor surface can be calculated by

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n},$$

where $\frac{\partial V}{\partial n}$ is the normal derivative of V at the surface. In this case the normal direction is the z-direction, so

$$\sigma = -\varepsilon_0 \left.\frac{\partial V}{\partial z}\right|_{z=0}$$

$$\Rightarrow \frac{\partial V}{\partial z} = \frac{1}{4\pi\varepsilon_0}\left\{\frac{-q(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{q(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}}\right\}$$

$$\Rightarrow \sigma(x,y) = \frac{-qd}{2\pi\left(x^2+y^2+d^2\right)^{3/2}}$$

As expected, the induced charge is negative (assuming q is positive) and greatest at $x = y = 0$.

The total induced charge $Q = \int \sigma da$.

This integral, over the xy-plane, could be done in Cartesian coordinates, with $da = dx\,dy$, but its easier to use polar coordinates (r,ϕ), with $r^2 = x^2 + y^2$ and $da = r dr d\phi$.

Then

$$\sigma(R) = \frac{-qd}{2\pi\left(r^2+d^2\right)^{3/2}}$$

and $Q = \int_0^{2\pi}\int_0^{\infty} \frac{-qd}{2\pi\left(r^2+d^2\right)^{3/2}} r dr d\phi = \left.\frac{qd}{\sqrt{r^2+d^2}}\right|_0^{\infty} = -q$

1.11.3 Force and Energy

The charge q is attracted towards the plane, because of the negative induced surface charge. The force:

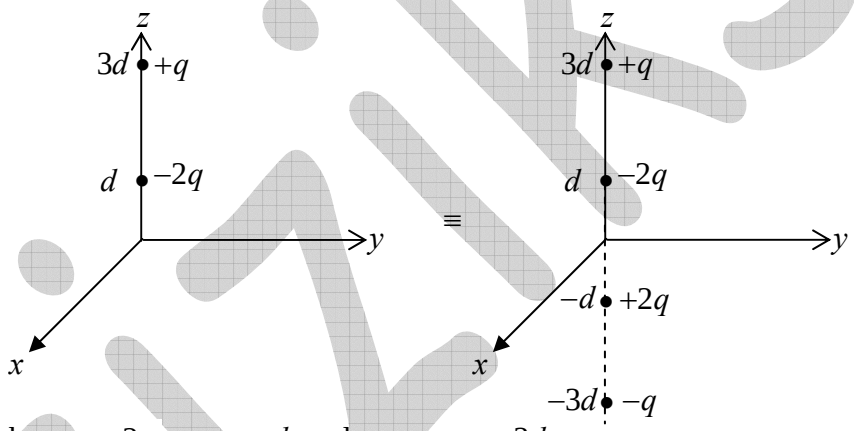
$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}.$$

One can determine the energy by calculating the work required to bring q in from infinity.

$$W = \int_{\infty}^{d} \vec{F}.d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{4z^2}dz = \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{4z}\right)\Big|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{4d}$$

Example: Find the force on the charge $+q$ as shown in figure (The $xy-$ plane is a grounded conductor).

Solution:



Place image charges $+2q$ at $z=-d$ and $-q$ at $z=-3d$.

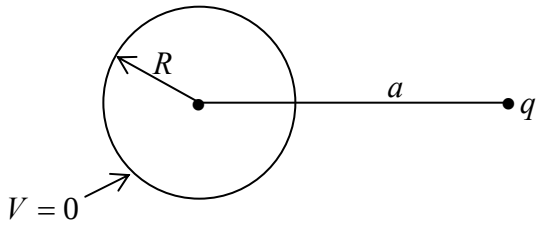
Total force on $+q$ is

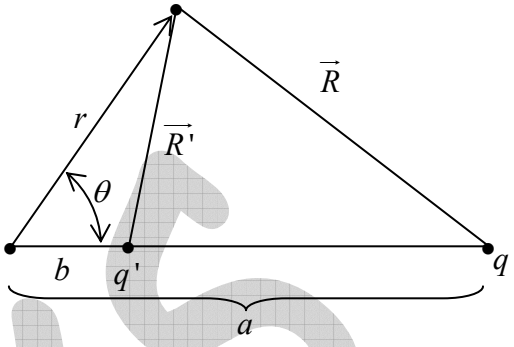
$$\vec{F} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-2q}{(2d)^2}+\frac{2q}{(4d)^2}+\frac{-q}{(6d)^2}\right]\hat{z} = -\frac{1}{4\pi\varepsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}$$

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}$$

1.11.4 Other Image Problem

The method just described is not limited to a single point charge; any stationary charge distribution near a grounded conducting plane can be treated in the same way, by introducing its mirror image.





Let us examine the completely different configuration, consisting of the point charge q together with another point charge

$$q' = -\frac{R}{a}q$$

placed at a distance

$$b = \frac{R^2}{a}$$

to the right of the centre of sphere. No conductor, now-just two point charges. The potential of this configuration is

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R} + \frac{q'}{R'}\right)$$

where $R = \sqrt{r^2 + a^2 - 2ra\cos\theta}$, $\quad R' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$

$$\Rightarrow V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left\{\frac{q}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} + \frac{q'}{\sqrt{r^2 + b^2 - 2rb\cos\theta}}\right\}$$

$$V(r,\theta) = \frac{q}{4\pi\varepsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right\}$$

Clearly when $r = R, V \to 0$

Induced charge

$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$. In this case $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$.

$$\Rightarrow \sigma(\theta) = -\varepsilon_0 \left.\frac{\partial V(r,\theta)}{\partial r}\right|_{r=R}$$

$$= -\frac{q}{4\pi}\left\{-(R^2+a^2-2Ra\cos\theta)^{-3/2}(R-a\cos\theta)+(R^2+a^2-2Ra\cos\theta)^{-3/2}\left(\frac{a^2}{R}-a\cos\theta\right)\right\}$$

$$= \frac{q}{4\pi}(R^2+a^2-2Ra\cos\theta)^{-3/2}\left[R-a\cos\theta-\frac{a^2}{R}+a\cos\theta\right]$$

$$\sigma(\theta) = \frac{q}{4\pi R}(R^2-a^2)(R^2+a^2-2Ra\cos\theta)^{-3/2}$$

$$q_{included} = q' = \int \sigma da = \frac{q}{4\pi R}(R^2-a^2)\int (R^2+a^2-2Ra\cos\theta)^{-3/2} R^2 \sin\theta d\theta d\phi$$

$$q' = \frac{q}{4\pi R}(R^2-a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2+a^2-2Ra\cos\theta)^{-1/2}\right]_0^{\pi}$$

$$q' = \frac{q}{2a}(a^2-R^2)\left[\frac{1}{\sqrt{R^2+a^2+2Ra}}-\frac{1}{\sqrt{R^2+a^2-2Ra}}\right]$$

But $a > R$ (else q would be inside), so $\sqrt{R^2+a^2-2Ra} = a-R$

$$q' = \frac{q}{2a}(a^2-R^2)\left[\frac{1}{(a+R)}-\frac{1}{(a-R)}\right] = -\frac{q}{a}R$$

$$\Rightarrow q' = -\frac{q}{a}R$$

Force

The force on q, due to the sphere, is the same as the force of the image charge q', thus:

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\varepsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{\left(a-\frac{R^2}{a}\right)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

Energy

To bring q in from infinity to a, we do work

$$W = \frac{q^2 R}{4\pi\varepsilon_0}\int_{\infty}^{a} \frac{\bar{a}}{(\bar{a}^2 - R^2)^2}\, d\bar{a} = \frac{q^2 R}{4\pi\varepsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2 - R^2)}\right]_{\infty}^{a} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2 R}{2(a^2 - R^2)}$$

$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2 R}{2(a^2 - R^2)}$$

fiziks

MCQ (Multiple Choice Questions)

Q1. Two point charges $Q_1 = 1nC$ and $Q_2 = 2nC$ are kept in free space such that the distance between them is $0.1m$.

(a) The force on Q_2 is along the direction from Q_2 to Q_1.

(b) The force on Q_2 is the same in magnitude as that on Q_1.

(c) The force on Q_1 is attractive.

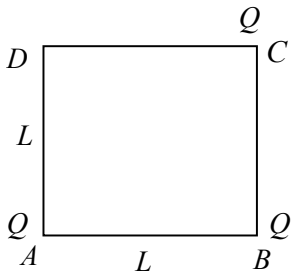
(d) A point charge $Q_3 = -3nC$, placed at the midpoint between Q_1 and Q_2, experiences no net force.

Q2. A ring of radius R carries a uniformly distributed charge $+Q$. A point charge $-q$ is placed on the axis of the ring at a distance x $(x << R)$ from the centre of the ring and released from rest. Then particle executes:

(a) a S.H.M. along the axis of ring (b) a S.H.M. along the circular path

(c) a motion along parabolic path (d) a motion along a hyperbolic path

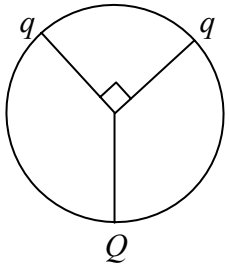
Q3. Three charges, each equal to Q, are placed at the three corners $(A, B \text{ and } C)$ of a square of side L. Then the magnitude of electric field at the fourth corner D is:



(a) $\dfrac{Q}{2\sqrt{2}\pi\varepsilon_0 L^2}$ (b) $\left(2\sqrt{2}+1\right)\dfrac{Q}{8\pi\varepsilon_0 L^2}$

(c) $\left(2\sqrt{2}+1\right)\dfrac{Q}{8\pi\varepsilon_0 L^3}$ (d) $\left(2\sqrt{2}-1\right)\dfrac{Q}{8\pi\varepsilon_0 L^2}$

Q4. Three charges are located on the circumference of a circle of radius R as shown in the figure below. The two charges q subtends an angle 90° at the centre of the circle. The charge Q is symmetrically placed with respect to the charges q. If the electric field at the centre of the circle is zero, what is the magnitude of Q?



(a) $q/\sqrt{2}$ (b) $\sqrt{2}q$

(c) $2q$ (d) $4q$

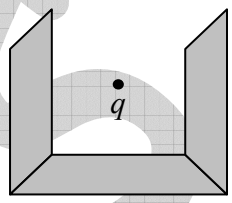
Q5. A loop of diameter d is rotated in a uniform electric field until the position of maximum electric flux is found. The flux in this position is measured to be ϕ. What is the electric field strength?

(a) $\frac{4\phi}{\pi d^2}$ (b) $\frac{2\phi}{\pi d^2}$ (c) $\frac{\phi}{\pi d^2}$ (d) $\frac{\pi\phi d^2}{4}$

Q6. A charge q is placed symmetrically with respect to three faces of equal area as shown in the figure. What is the total electric flux through the faces?

(a) $\frac{q}{\varepsilon_0}$ (b) $\frac{q}{2\varepsilon_0}$

(c) $\frac{q}{3\varepsilon_0}$ (d) zero



Q7. The electric field in a certain region of space is given by $\vec{E} = 200\hat{x}V/m$. Which of the following is a correct combination of the flux passing through an area A (in square metre) lying in

(i) x–y plane, (ii) x–z plane & (iii) y–z plane respectively?

(a) 0, 200 A, 0 (b) 200 A, 0, 0

(c) 0, 0, 200 A (d) 0, 0, 0

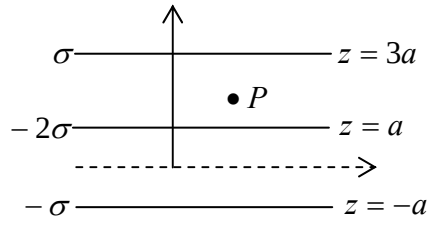
Q8. A sphere of radius R carries charge density ρ proportional to the square of distance from the center such that $\rho = Ar^2$, where A is a positive constant. At a distance $R/2$ from the center, the magnitude of electric field is

(a) $\frac{A}{4\varepsilon_0}$ (b) $\frac{AR^3}{40\varepsilon_0}$ (c) $\frac{AR^3}{24\varepsilon_0}$ (d) $\frac{AR^3}{5\varepsilon_0}$

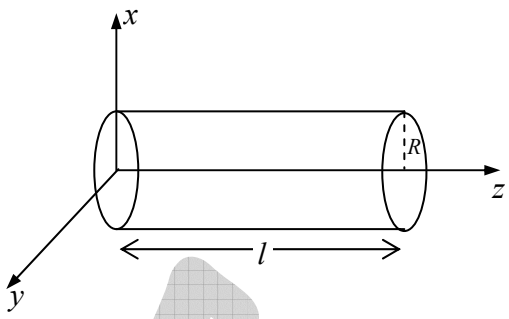
Q9. Three infinitely long charge sheets are placed as shown in figure. The electric field at point P is

(a) $\frac{2\sigma}{\varepsilon_0}\hat{z}$ (b) $\frac{4\sigma}{\varepsilon_0}\hat{z}$

(c) $-\frac{2\sigma}{\varepsilon_0}\hat{z}$ (d) zero



Q10. A cylindrical rod of length l and radius R, made of an inhomogeneous dielectric, is placed with its axis along the z-direction with one end at the origin as shown in figure below. If the volume charge density $\rho = -10z$, then the total charge inside the dielectric is



(a) zero (b) $10\pi R^2 l$

(c) $-5\pi R^2 l$ (d) $-5\pi R^2 l^2$

Q11. Given that electric field $\vec{E} = \frac{Q}{\pi\varepsilon_0 r^2}(1-\cos 3r)\hat{r}$ in spherical polar coordinates. The charge density associated with it is:

(a) $\frac{Q}{2\pi\varepsilon_0 r^2}\left(\frac{-2}{r^3}+3\sin 3r\right)$ (b) $\frac{3Q}{\pi\varepsilon_0 r^2}\sin 3r$

(c) $\frac{3Q}{\pi r^2}\sin 3r$ (d) $\frac{3Q}{\pi\varepsilon_0 r^2}\cos 3r$

Q12. The charge density in spherical polar coordinate is $\rho = -\frac{\varepsilon_0 Ab}{r^2}e^{-br} + 4\pi\varepsilon_0 A\delta(r)$. Then the total charge contained in a sphere of infinite radius is: $\left(use \int_0^\infty \delta(\vec{r})4\pi r^2 dr = 1\right)$

(a) 0 (b) $4\pi\varepsilon_0 A$ (c) $-4\pi\varepsilon_0 A$ (d) $\varepsilon_0 Abe^{-br}$

Q13. A static charge distribution gives rise to an electric field of the form $\vec{E} = \alpha e^{-r/R}\frac{\hat{r}}{r^2}$, where α and R are positive constants. The charge contained within a sphere of radius R, centred at the origin is

(a) $\pi\alpha\varepsilon_0\frac{e}{R^2}$ (b) $\pi\alpha\varepsilon_0\frac{e^2}{R^2}$ (c) $4\pi\alpha\varepsilon_0\frac{R}{e}$ (d) $\frac{4\pi\alpha\varepsilon_0}{e}$

Q14. A sphere of radius $2R$ has a uniform charge density ρ. The difference in electrostatic potential at $r = R$ and $r = 0$ is: (where r is the distance from center)

(a) $-\frac{\rho R^2}{\varepsilon_0}$ (b) $-\frac{2\rho R^2}{\varepsilon_0}$ (c) $-\frac{\rho}{6\varepsilon_0 R}$ (d) $-\frac{\rho R^2}{6\varepsilon_0}$

Q15. A charge Q is distributed over two concentric hollow spheres of radii r and R $(R>r)$ such that the surfaces charge densities are equal. Then the potential at the common center is

(a) $\dfrac{Q}{4\pi\varepsilon_0(r+R)}$ (b) $\dfrac{Q(r+R)}{4\pi\varepsilon_0(r^2-R^2)}$

(c) $\dfrac{Q(r+R)}{4\pi\varepsilon_0(r^2+R^2)}$ (d) $\dfrac{Q(r-R)}{4\pi\varepsilon_0(r^2-R^2)}$

Q16. The plates of a parallel plate capacitor (which are normal to the x-axis) are located at $x=0$ and $x=L$. The plate at $x=0$ is grounded while the other plate is at a potential V_0. The space between the plates has uniform volume charge density ρ. The potential $V(x)$ between the plates is given by

(a) $-\dfrac{\rho}{2\varepsilon_0}x^2+\left(\dfrac{V_0}{L}+\dfrac{\rho L}{2\varepsilon_0}\right)x$ (b) $\dfrac{\rho}{2\varepsilon_0}x^2-\left(\dfrac{V_0}{L}+\dfrac{\rho L}{2\varepsilon_0}\right)x$

(c) $-\dfrac{\rho}{2\varepsilon_0}x^2-\left(\dfrac{V_0}{L}+\dfrac{\rho L}{2\varepsilon_0}\right)x$ (d) $\dfrac{\rho}{2\varepsilon_0}x^2+\left(\dfrac{V_0}{L}+\dfrac{\rho L}{2\varepsilon_0}\right)x$

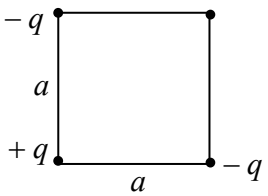
Q17. If the electrostatic potential in spherical polar coordinates is

$$\phi(r)=\phi_0 e^{-r/r_0}$$

where ϕ_0 and r_0 are constants, then the charge density at a distance $r=r_0$ will be

(a) $\dfrac{\varepsilon_0\phi_0}{er_0^2}$ (b) $\dfrac{e\varepsilon_0\phi_0}{2r_0^2}$ (c) $-\dfrac{\varepsilon_0\phi_0}{er_0^2}$ (d) $-\dfrac{2e\varepsilon_0\phi_0}{r_0^2}$

Q18. Three charges are situated at the corners of a square (side a) as shown in figure. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corner?

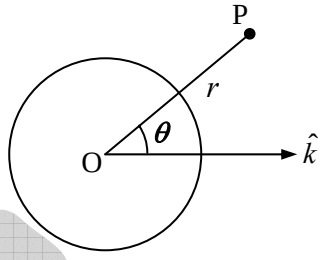


(a) $\dfrac{q^2}{4\pi\varepsilon_0 a}\left(2-\dfrac{1}{\sqrt{2}}\right)$ (b) $\dfrac{q^2}{4\pi\varepsilon_0 a}\left(-2+\dfrac{1}{\sqrt{2}}\right)$

(c) $\dfrac{q}{4\pi\varepsilon_0 a}\left(2-\dfrac{1}{\sqrt{2}}\right)$ (d) $\dfrac{q}{4\pi\varepsilon_0 a}\left(-2+\dfrac{1}{\sqrt{2}}\right)$

Q19. A spherical conductor of radius a is placed in a uniform electric field $\vec{E} = E_0\hat{k}$. The potential at a point $P(r,\theta)$ for $r > a$, is given by

$$\Phi(r,\theta) = constant - E_0 r\sin\theta + \frac{E_0 a^3}{r^2}\sin\theta$$



where r is the distance of P from the centre O of the sphere and θ is the angle OP makes with the z-axis. The charge density on the sphere at $\theta = 30^0$ is

(a) $3\sqrt{3}\varepsilon_0 E_0/2$ (b) $3\varepsilon_0 E_0/2$ (c) $\sqrt{3}\varepsilon_0 E_0/2$ (d) $\varepsilon_0 E_0/2$

Q20. Eight charges $+q$ and $-q$ are located alternatively at the corners of a cube of side a. The work done to separate the charges to infinite distance is:

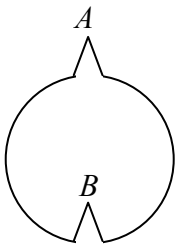
(a) $\frac{q^2}{\pi\varepsilon_0 a}\left(3\sqrt{6}+3\sqrt{3}+\sqrt{2}\right)$ (b) $\frac{q^2}{\pi\varepsilon_0 a}\left(-3\sqrt{6}+3\sqrt{3}-\sqrt{2}\right)$

(c) $\frac{q^2}{\pi\varepsilon_0 a}\left(3+\frac{3}{\sqrt{2}}\right)$ (d) $\frac{q^2}{\pi\varepsilon_0 a}\left(3+\frac{1}{\sqrt{3}}\right)$

Q21. A point charge q moves from point P to point S along the path $PQRS$ (as shown in figure) in a uniform electric field $\vec{E}$ pointing parallel to the positive direction of the X-axis. The coordinates P, Q, R and S are $(a,b,0), (2a,0,0)$, $(-a,b,0)$ and $(0,0,0)$ respectively. The work done by the field in the above process is

(a) *zero* (b) qEa^2 (c) $-qEa$ (d) $-qEa^2$

Q22. A solid spherical conductor has a conical hole made at one end, ending in a point B, and a small conical projection of the same shape and size at the opposite side, ending in a point A. A cross-section through the centre of the conductor is shown in the figure on the right. If, now, a positive charge Q is transferred to the sphere, then



(a) $\sigma_A = \sigma_B$ (b) $\sigma_A > \sigma_B$ (c) $\sigma_A < \sigma_B$ (d) $\sigma_A = 2\sigma_B$

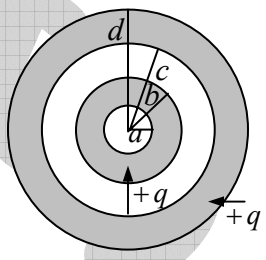
Q23. Consider two concentric conducting spherical shells with inner and outer radii a, b and c, d as shown in the figure. Both the shells are given q amount of positive charges. The electric fields in different regions are

(a) $\vec{E} = 0$ for $r < a$; $\vec{E} = \frac{-q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $a < r < b$.

$\vec{E} = 0$ for $b < r < c$; $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > d$.

(b) $\vec{E} = \frac{-q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r < a$; $\vec{E} = 0$ for $a < r < b$.

$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $b < r < c$; $\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > d$.



(c) $\vec{E} = \frac{-q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r < a$; $\vec{E} = 0$ for $a < r < b$.

$\vec{E} = 0$ for $b < r < c$; $\vec{E} = \frac{2q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > d$.

(d) $\vec{E} = 0$ for $r < a$; $\vec{E} = 0$ for $a < r < b$.

$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $b < r < c$; $\vec{E} = \frac{2q}{4\pi\varepsilon_0 r^2}\hat{r}$ for $r > d$.

Q24. A "pure" dipole with dipole moment $\vec{p} = p_o\hat{z}$ is situated at the origin. A point charge Q is moved from the point (a, 0, 0) to (0, 0, a) then the work done will be

(a) *zero* (b) $\frac{p_0 Q}{4\pi\varepsilon_0 a^3}$ (c) $\frac{p_0}{4\pi\varepsilon_0 a^2}$ (d) $\frac{p_0 Q}{4\pi\varepsilon_0 a^2}$

Q25. Four point charges are placed in a plane at the following positions:

$+Q$ at (1, 0), $-Q$ at (–1, 0) $+Q$ at (0, 1) and $-Q$ at (0, –1).

At large distances the electrostatic potential due to this charge distribution will be dominated by the

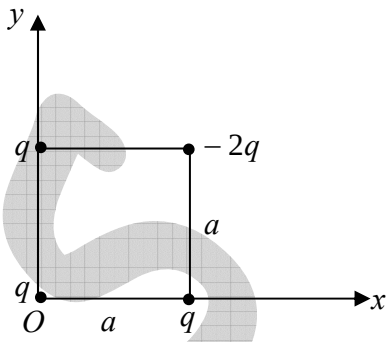
(a) Monopole moment (b) Dipole moment

(c) Quadrupole moment (d) Octopole moment

Q26. Three point charges q, q and $-2q$ are located at $(0,-a, a)$, $(0, a, a)$ and $(0, 0,-a)$ respectively. The net dipole moment of this charge distribution is

(a) $4qa\hat{k}$ (b) $2qa\hat{k}$ (c) $-4qa\hat{i}$ (d) $-2qa\hat{j}$

Q27. Four charges are placed at the four corners of a square of side a as shown in the figure. The electric dipole moment of this configuration is



(a) $\vec{p} = qa\hat{i} + qa\hat{j}$

(b) $\vec{p} = -qa\hat{i} + qa\hat{j}$

(c) $\vec{p} = -qa\hat{i} - qa\hat{j}$

(d) $\vec{p} = qa\hat{i} - qa\hat{j}$

Q28. A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is measured from the centre of the sphere. The bound surface and volume charge densities are given, respectively, by

(a) $-k|\vec{r}|$ and $3k$ (b) $k|\vec{r}|$ and $-3k$

(c) $k|\vec{r}|$ and $-4\pi kr$ (d) $k|\vec{r}|$ and $4\pi kr$

Q29. A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is measured from the centre of the sphere. The electric field $\vec{E}$ at a point $\vec{r}$ outside the sphere is given by

(a) $\vec{E} = 0$ (b) $\vec{E} = \dfrac{kR(R^2 - r^2)}{\varepsilon_0 r^3}\hat{r}$

(c) $\vec{E} = \dfrac{kR(R^2 - r^2)}{\varepsilon_0 r^5}\hat{r}$ (d) $\vec{E} = \dfrac{-kr}{\varepsilon_0}\hat{r}$

Q30. A dielectric sphere of radius R, permittivity ε is charged throughout its volume by a charge density $\rho = \dfrac{A}{r}$, where A is some constant and r being the distance from the center of the sphere. Then the electrostatic energy of the system is proportional to:

(a) R^2 (b) R^3 (c) R^4 (d) R^5

Q31. A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is measured from the centre of the sphere. The electric field $\vec{E}$ at a point $\vec{r}$ inside the sphere is given by

(a) $\vec{E} = 0$

(b) $\vec{E} = \dfrac{kR(R^2 - r^2)}{\varepsilon_0 r^3}\hat{r}$

(c) $\vec{E} = \dfrac{kR(R^2 - r^2)}{\varepsilon_0 r^5}\hat{r}$

(d) $\vec{E} = \dfrac{-kr}{\varepsilon_0}\hat{r}$

Q32. A spherical conductor of radius R_1 carries a charge Q. It is surrounded by linear dielectric material of relative permittivity ε_r, out to radius R_2. The energy of this configuration is:

(a) $\dfrac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left[\dfrac{1}{R_1} + \dfrac{\varepsilon_r - 1}{R_2}\right]$ (b) $\dfrac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left[\dfrac{1}{R_1} + \dfrac{\varepsilon_r}{R_2}\right]$

(c) $\dfrac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left[\dfrac{\varepsilon_r - 1}{R_1} + \dfrac{1}{R_2}\right]$ (d) zero.

NAT (Numerical Answer Type)

Q33. A uniform line charge, infinite in extent, having charge per unit length $20\ nc/m$ lies along the z-axis. The electric field $\vec{E}$ at $(6,8,3)m$ is……$\hat{r}\ V/m$

Q34. The electric field at 20 cm from the centre of a uniformly charged dielectric sphere of radius 10 cm is E. Then, at a distance 5 cm from the center, it will be……E

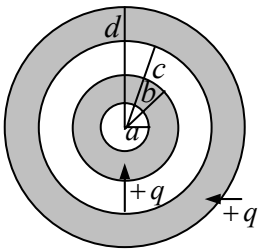
Q35. The energy stored in a uniformly charged solid sphere of radius R and charge q is $\dfrac{\alpha}{4\pi\varepsilon_0}\dfrac{q^2}{R}$. Then the value of α is …….

Q36. Two electric dipoles P_1 and P_2 are placed at $(0,0,0)$ and $(a,0,0)$ respectively with both of them pointing in the $+z$ direction. Without changing the orientations of the dipoles P_2 is moved to $(0,2a,0)$. The ratio of the electrostatic potential energy of the dipoles before moving to that after moving is……

Q37. Four equal point charges are kept fixed at the four vertices of a square. How many neutral points (i.e. points where the electric field vanishes) will be found inside the square?

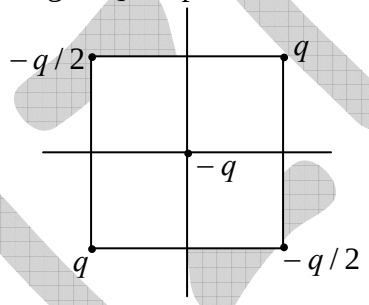
Q38. If the electrostatic potential at a point (x, y) is given by $V = (2x + 4y)$ volts, the electrostatic energy density at that point is…… $\varepsilon_0 J/m^3$

Q39. Consider two concentric conducting spherical shells with inner and outer radii a, b and c, d as shown in the figure. Both the shells are given q amount of positive charges. In order to have equal surface charge densities on the outer surface of both the shells, the ratio $\frac{d}{b}$ is ……



Q40. The electric field at a point due to an electric dipole is perpendicular to the dipole axis, the angle between the dipole axis and the line joining the point with the centre of the dipole is $\tan^{-1}(\beta)$. Then the value of β is ……..

Q41. Let four point charges $q, -q/2, q$ and $-q/2$ be placed at the vertices of a square of side a. Let another point charge $-q$ be placed at the cnetre of the square (see the figure).



Let $V(r)$ be the electrostatic potential at a point P at a distance $r >> a$ from the centre of the square. Then $V(3r)/V(r)$ is……

MSQ (Multiple Select Questions)

Q42. For given electrostatic field $\vec{E} = k\left[y^2\hat{x} + \left(2xy + z^2\right)\hat{y} + 2yz\hat{z} \right]$, which of the following does not represent the possible potential for the given field (use the origin as your reference point?)

(a) $V(x,y,z) = -k\left(xy^2 + yz\right)$ (b) $V(x,y,z) = -k\left(xy + yz^2\right)$

(c) $V(x,y,z) = k\left(xy^2 + yz^2\right)$ (d) $V(x,y,z) = -k\left(xy^2 + yz^2\right)$

Q43. Which of the following statements regarding the electric fields $\vec{E}_1 = x\hat{x} + y\hat{y}$ and $\vec{E}_2 = xy^2\hat{x} + y^2\hat{y}$ are **NOT** correct?

(a) Both $\vec{E}_1$ and $\vec{E}_2$ represents an electrostatic field

(b) Neither $\vec{E}_1$ nor $\vec{E}_2$ can represent electrostatic field

(c) Only $\vec{E}_1$ can represent electrostatic field

(d) Only $\vec{E}_2$ can represent an electrostatic field

Q44. If the electrostatic potential were given by $\phi = \phi_0(x^2 + y^2 + z^2)$, where ϕ_0 is constant then which of the following statements are correct

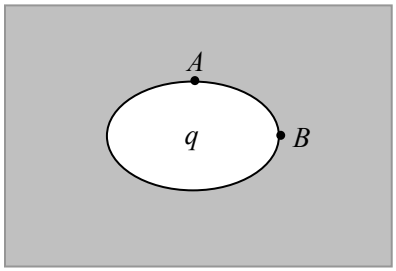
(a) The electric fields $\vec{E} = 2\phi_0\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$

(b) The electric fields $\vec{E} = -2\phi_0\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$

(c) Charge density $\rho = -2\phi_0\varepsilon_0$

(d) Charge density $\rho = -6\phi_0\varepsilon_0$

Q45. An ellipsoidal cavity is carved within a perfect conductor as shown in figure. A Positive charge '*q*' is placed at the center of the cavity. The points *A* & *B* are on the cavity surface as shown in figure. Then which of the following statements are true?



(a) Electric field near *A* in the cavity $\neq$ electric field near *B* in the cavity

(b) Charge density at *A* $\neq$ charge density at *B*

(c) Potential at *A* = Potential at *B*

(d) Total electric flux through the surface of the cavity is $\frac{q}{\varepsilon_0}$

Q46. A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius R). Then which of the following statements are true?

(a) The total bound charge on the surface is $\frac{q\chi_e}{1+\chi_e}$

(b) The total bound charge on the surface is $-\frac{q\chi_e}{1+\chi_e}$

(c) The total volume bound charge $-\frac{q\chi_e}{1+\chi_e}$

(d) The total volume bound charge $\frac{q\chi_e}{1+\chi_e}$

Q47. A sphere of radius R carries a polarization $\vec{P} = k\vec{r}$, where k is a constant and $\vec{r}$ is measured from the centre of the sphere. Then which of the following are true for electric field $\vec{E}$:

(a) Electric field inside the sphere is $\vec{E} = 0$

(b) Electric field inside the sphere is $\vec{E} = \frac{-kr}{\varepsilon_0}\hat{r}$

(c) Electric field outside the sphere is $\vec{E} = 0$

(d) Electric field outside the sphere is $\vec{E} = \frac{-kr}{\varepsilon_0}\hat{r}$

Q48. A charge Q is uniformly distributed in spherical linear dielectric material. The radius of the sphere is R and its dielectric constant is ε_r. Which of the following statements are true

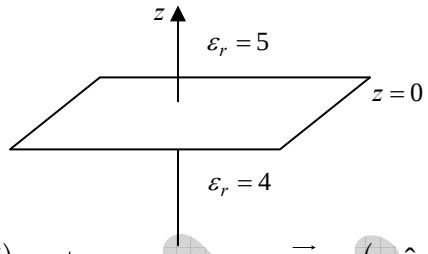
(a) The electric $\vec{E} = \frac{\rho r}{3\varepsilon_0\varepsilon_r}\hat{r}$ for $r < R$

(b) The electric $\vec{E} = \frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}$ for $r > R$

(c) The potential at the center of sphere is $\frac{Q}{4\pi\varepsilon_0 R}\left(1+\frac{1}{2\varepsilon_r}\right)$

(d) The potential at the center of sphere is $\frac{Q}{4\pi R}\left(1+\frac{1}{3\varepsilon_0\varepsilon_r}\right)$

Q49. Assume that $z = 0$ plane is the interface between two linear and homogenous dielectrics (see figure). The relative permittivities are $\varepsilon_r = 5$ for $z > 0$ and $\varepsilon_r = 4$ for $z < 0$. The electric field in the region $z > 0$ is $\vec{E}_1 = \left(3\hat{i} - 5\hat{j} + 4\hat{k}\right) kV/m$. If there are no free charges on the interface, then which of the following is true for the electric field in the region $z < 0$ is given by



(a) $\vec{D}_2 = \left(12\hat{i} - 20\hat{j} + 20\hat{k}\right) kV/m$

(b) $\vec{D}_2 = \left(15\hat{i} - 25\hat{j} + 25\hat{k}\right) kV/m$

(c) $\vec{E}_2 = \left(3\hat{i} - 5\hat{j} - 5\hat{k}\right) kV/m$

(d) $\vec{E}_2 = \left(3\hat{i} - 5\hat{j} + 5\hat{k}\right) kV/m$

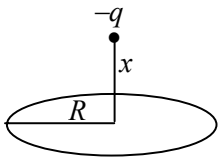
Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (a)

Solution:



Field along the axis of ring $E=\frac{1}{4\pi\varepsilon_0}\frac{Qx}{\left(R^2+x^2\right)^{3/2}} \Rightarrow E=\frac{1}{4\pi\varepsilon_0}\frac{Qx}{R^3}$ since $x<<R$

$\Rightarrow F=-qE=-kx$ (Charge will execute S.H.M.)

Ans. 3: (b)

Solution: Electric field due to charge at A, $E_A=\frac{Q}{4\pi\varepsilon_0 L^2}$ along AD

Electric field due to charge at C, $E_C=\frac{Q}{4\pi\varepsilon_0 L^2}$ along CD

Electric field due to charge at D, $E_D=\frac{Q}{4\pi\varepsilon_0\left(\sqrt{2}L\right)^2}$ along BD

Thus resultant field $E=\sqrt{E_A^2+E_C^2}+E_B=\left(2\sqrt{2}+1\right)\frac{Q}{8\pi\varepsilon_0 L^2}$

Ans. 4: (b)

Solution: $E_1=E_2=\frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}$ and $E_3=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2}$ (upward)

Resultant of E_1 *and* E_2 is $E=\sqrt{E_1^2+E_2^2}=\sqrt{2}E_1$ (downward)

Thus for $E_3=E\Rightarrow Q=\sqrt{2}q$

Ans. 5: (a)

Solution: Maximum flux $\phi=EA=E\pi\left(\frac{d}{2}\right)^2=E\pi\frac{d^2}{4}\Rightarrow E=\frac{4\phi}{\pi d^2}$

Ans. 6: (b)

Solution: Electric flux through a cube (6 faces) when charge is at its center $= \frac{q}{\varepsilon_0}$

Flux through three faces $= \frac{q}{2\varepsilon_0}$

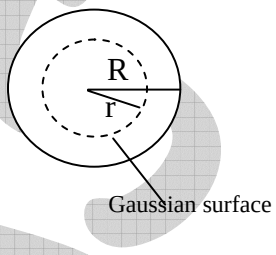
Ans. 7: (c)

Solution: $Electric\ flux = \vec{E}.\vec{A} = (200\hat{x}).(A\hat{x}) = 200A$.

Ans. 8: (b)

$$\oint \vec{E}d\vec{s} = \frac{Q_{enc}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int \rho d\tau \Rightarrow |\vec{E}|.4\pi r^2 = \frac{1}{\varepsilon_0}\int Ar^2.4\pi r^2 dr = \frac{4\pi A r^5}{5\varepsilon_0}$$

$$|\vec{E}| = \frac{Ar^3}{5\varepsilon_0} \Rightarrow \ at\ \ r = \frac{R}{2}, \quad |\vec{E}| = \frac{AR^3}{40\varepsilon_0}$$



Ans. 9: (c)

Solution: $\vec{E}_P = \vec{E}_\sigma + \vec{E}_{-2\sigma} + \vec{E}_{-\sigma} = \frac{\sigma}{2\varepsilon_0}(-\hat{z}) + \frac{2\sigma}{2\varepsilon_0}(-\hat{z}) + \frac{\sigma}{2\varepsilon_0}(-\hat{z}) \Rightarrow \vec{E}_P = \frac{-2\sigma}{\varepsilon_0}\hat{z}$

Ans. 10: (d)

Solution: $q_{inside} = \int \rho d\tau = \int\int\int \rho r dr d\phi dz = \int_0^l (-10z)\pi R^2 dz = -5\pi R^2 l^2$

Ans. 11: (c)

Solution: $\rho = \varepsilon_0(\vec{\nabla}.\vec{E}) = \varepsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{Q}{\pi\varepsilon_0 r^2}(1-\cos 3r)\right] = \frac{3Q}{\pi r^2}\sin 3r$

Ans. 12: (a)

Solution: The total charge is

$$Q = \int_{all\ space} \rho d\tau = -\int_0^\infty \frac{\varepsilon_0 A b e^{-br}}{r^2} 4\pi r^2 dr + \int_{all\ space} 4\pi\varepsilon_0 A\delta(r)\, d\tau$$

$$\Rightarrow Q = 4\pi\varepsilon_0 A\left[e^{-br}\right]_0^\infty + 4\pi\varepsilon_0 A \Rightarrow Q = -4\pi\varepsilon_0 A + 4\pi\varepsilon_0 A = 0$$

Ans. 13: (d)

Solution: $Q_{enc} = \varepsilon_0 \oint \vec{E} \cdot d\vec{a} = -\alpha\varepsilon_0 \int e^{-r/R} \frac{\hat{r}}{r^2} \cdot \left(r^2 \sin\theta d\theta d\phi \hat{r}\right) = -\alpha\varepsilon_0 \times \int_0^{\pi}\int_0^{2\pi} e^{-r/R} \sin\theta d\theta d\phi$

at $r = R$, $Q_{enc} = \frac{4\pi\alpha\varepsilon_0}{e}$

Ans. 14: (d)

Solution: $V(R) - V(0) = -\int_0^R \vec{E}.d\vec{l} = -\int_0^R \frac{\rho r}{3\varepsilon_0} dr = -\frac{\rho R^2}{6\varepsilon_0}$

Ans. 15: (c)

Solution: Let charge on inner and outer shell be q_1 and q_2 then $Q = q_1 + q_2$

Potential at common centre is $V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{r} + \frac{q_2}{R}\right) = \frac{1}{4\pi\varepsilon_0} q_2 \left(\frac{q_1}{q_2 r} + \frac{1}{R}\right)$

Since $\sigma_1 = \sigma_2 \Rightarrow \frac{q_1}{4\pi r^2} = \frac{q_2}{4\pi R^2} \Rightarrow \frac{q_1}{q_2} = \frac{r^2}{R^2}$

$\Rightarrow \frac{q_1 + q_2}{q_2} = \frac{r^2 + R^2}{R^2} \Rightarrow q_2 = \frac{QR^2}{r^2 + R^2}$ and $q_1 = \frac{Qr^2}{r^2 + R^2} \Rightarrow V = \frac{Q(r+R)}{4\pi\varepsilon_0\left(r^2 + R^2\right)}$

Ans. 16: (a)

Solution: The Laplace's equation in Cartesian coordinates system is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$

as V is only function of x, we have the differential equation, $\frac{d^2 V}{dx^2} = -\frac{\rho}{\varepsilon_0}$

by integrating we have the solution of this equation as

$\frac{dV}{dx} = -\frac{\rho}{\varepsilon_0} x + A \Rightarrow V(x) = -\frac{\rho}{2\varepsilon_0} x^2 + Ax + B$ where A and B are constants.

The two equations need to be solved for the following boundary conditions:

(i) $x = 0; V = 0$ (ii) $x = L; V = V_0$

Substituting these boundary conditions, we get

$$\text{At } x=0,\ V(0)=0=0+0+B\Rightarrow B=0$$

$$\text{At } x=L,\ V(L)=V_0=-\frac{\rho}{2\varepsilon_0}L^2+AL\Rightarrow A=\frac{V_0}{L}+\frac{\rho L}{2\varepsilon_0}\Rightarrow V(x)=-\frac{\rho}{2\varepsilon_0}x^2+\left(\frac{V_0}{L}+\frac{\rho L}{2\varepsilon_0}\right)x$$

Ans. 17: (a)

Solution: $\because \nabla^2\phi=-\frac{\rho}{\varepsilon_0}\Rightarrow \rho=-\varepsilon_0\left(\nabla^2\phi\right)$

$$\nabla^2\phi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\times-\frac{\phi_0}{r_0}e^{-r/r_0}\right)=-\frac{1}{r^2}\frac{\phi_0}{r_0}\frac{\partial}{\partial r}\left(r^2\times e^{-r/r_0}\right)$$

$$=-\frac{1}{r^2}\frac{\phi_0}{r_0}\left[r^2\times-\frac{1}{r_0}e^{-r/r_0}+2re^{-r/r_0}\right]\Rightarrow\nabla^2\phi=-\frac{\phi_0}{r_0}\left[-\frac{1}{r_0}e^{-r/r_0}+\frac{2}{r}e^{-r/r_0}\right]$$

$$\text{At a distance } r=r_0,\ \nabla^2\phi=-\frac{\phi_0}{r_0}\left[\frac{1}{r_0}e^{-1}+\frac{2}{r_0}e^{-1}\right]=-\frac{\phi_0}{r_0^2e}\Rightarrow\rho=-\varepsilon_0\left(-\frac{\phi_0}{r_0^2e}\right)=\frac{\phi_0\varepsilon_0}{r_0^2e}$$

Ans. 18: (b)

Solution: Potential at fourth corner

$$V=\frac{1}{4\pi\varepsilon_0}\sum\frac{q_i}{r_i}=\frac{1}{4\pi\varepsilon_0}\left(-\frac{q}{a}+\frac{q}{\sqrt{2}a}-\frac{q}{a}\right)=\frac{q}{4\pi\varepsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

$$W=qV=\frac{q^2}{4\pi\varepsilon_0 a}\left(-2+\frac{1}{\sqrt{2}}\right)$$

Ans. 19: (b)

Solution: $\sigma=-\varepsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=a}=-\varepsilon_0\left[-E_0\sin\theta-\frac{2E_0a^3}{r^3}\sin\theta\right]_{r=a}$

$$\sigma=-\varepsilon_0\left[-E_0\sin\theta-2E_0\sin\theta\right]\Rightarrow\sigma=+3E_0\varepsilon_0\sin\theta=+3E_0\varepsilon_0\sin30^0=\frac{3}{2}\varepsilon_0E_0$$

Ans. 20: (b)

Solution: The charge pairs are ${}^8C_2=\frac{8\times7}{1\times2}=28$. Out of these 28 pairs, we have

(i) 12 pairs of dissimilar charges placed at a mutual separation of *a*. Thus

$$W_a = 12\frac{q\times -q}{4\pi\varepsilon_0 a} = -\frac{12q^2}{4\pi\varepsilon_0 a}\ .$$

(ii) 12 pairs of similar charges placed at a mutual separation of $\sqrt{2}a$. Thus

$$W_{\sqrt{2}a} = 12\frac{q\times q}{4\pi\varepsilon_0\times\sqrt{2}a} = \frac{12q^2}{4\sqrt{2}\pi\varepsilon_0 a}.$$

(iii) 4 pairs of dissimilar charges placed at a mutual separation of $\sqrt{3}a$. Thus

$$W_{\sqrt{3}a} = 4\frac{q\times -q}{4\pi\varepsilon_0\times\sqrt{3}a} = -\frac{4q^2}{4\sqrt{3}\pi\varepsilon_0 a}.$$

Total work done $W = W_a + W_{\sqrt{2}a} + W_{\sqrt{3}a} = \frac{q^2}{4\pi\varepsilon_0 a}\left(-12+\frac{12}{\sqrt{2}}-\frac{4}{\sqrt{3}}\right)$

$$\Rightarrow W = \frac{q^2}{4\pi\varepsilon_0 a}\left(-12+\frac{12}{\sqrt{2}}-\frac{4}{\sqrt{3}}\right) = \frac{q^2}{\pi\varepsilon_0 a}\left(-3\sqrt{6}+3\sqrt{3}-\sqrt{2}\right)$$

Ans. 21: (c)

Solution: Work done in a closed path in an electrostatic field is zero.

$$W_{PS}+W_{SP}=0 \Rightarrow W_{PS} = -W_{SP} = -\vec{F}.\vec{SP} = -\left(qE\hat{x}\right).\left(a\hat{x}+b\hat{y}\right) = -qEa^2$$

Ans. 22: (c)

Solution: $\sigma_A < \sigma_B$

Ans. 23: (d)

Ans. 24: (d)

Solution: $W = Q\left[V(0,0,a)-V(a,0,0)\right]$

$$V(r,\theta) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r^2} \Rightarrow V(0,0,a) = \frac{p_0}{4\pi\varepsilon_0 a^2} \quad \because \theta = 0 \text{ and } V(a,0,0) = 0 \quad \because \theta = \frac{\pi}{2}$$

$$\Rightarrow W = \frac{p_0 Q}{4\pi\varepsilon_0 a^2}$$

Ans. 25: (b)

Solution: $Q_{mono} = 0$, $\vec{p} = +Q\hat{i} - Q\left(-\hat{i}\right) + Q\hat{j} - Q\left(-\hat{j}\right) \neq 0$

Ans. 26: (a)

Solution: $\vec{p} = \left[q\times 0 + q\times 0 + \left(-2q\times 0\right)\right]\hat{i} + \left[q\times -a + q\times a + \left(-2q\times 0\right)\right]\hat{j}$

$$+\left[q\times a + q\times a + \left(-2q\times -a\right)\right]\hat{k}$$

$$\Rightarrow \vec{p} = 4qa\hat{k}$$

Ans. 27: (c)

Solution: $\vec{p} = \Sigma q_i \vec{r}_i' = q\times 0 + qa\hat{x} - 2q\left(a\hat{x} + a\hat{y}\right) + qa\hat{y} \Rightarrow \vec{p} = -qa\hat{x} - qa\hat{y}$

Ans. 28: (b)

Solution: $\sigma_b = \vec{P}.\hat{n} = (kR)\hat{r}.\hat{r} = kR$ at $r = R$ and $\rho_b = -\vec{\nabla}.\vec{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}(r^2.kr) = -3k$

Ans. 29: (a)

Solution: $\rho_b = -\vec{\nabla}.\vec{P} = -\left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{dy} + \hat{z}\frac{\partial}{dz}\right].\left(kx\hat{x} + ky\hat{y} + kz\hat{z}\right) = -k - k - k = -3k$

The electric field $\vec{E}$ at a point $\vec{r}$ outside the sphere is zero, since total bound charge is zero.

The electric field $\vec{E}$ at a point $\vec{r}$ inside the sphere is $\vec{E} = \frac{\rho_b r}{3\varepsilon_0}\hat{r} = \frac{-kr}{\varepsilon_0}\hat{r}$.

Ans. 30: (b)

Solution: $\vec{E} = \frac{AR^2}{2\varepsilon_0 r^2}\hat{r}$ when $r > R$ and $\vec{D} = \frac{A}{2}\hat{r}$ where $r < R$

Total energy of the system $W = \frac{1}{2}\int_{\text{all space}} \vec{D}.\vec{E}\,d\tau$

$$W = \frac{1}{2}\int_0^R \frac{A}{2}\times\frac{AR^2}{2\varepsilon_0 r^2}\times 4\pi r^2 dr + \frac{1}{2}\varepsilon_0\int_R^\infty\left(\frac{AR^2}{2\varepsilon_0 r^2}\right)^2\times 4\pi r^2 dr \Rightarrow W \propto R^3$$

Ans. 31: (d)

Solution: $\rho_b = -\vec{\nabla}.\vec{P} = -\left[\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{dy} + \hat{z}\frac{\partial}{dz}\right].\left(kx\hat{x} + ky\hat{y} + kz\hat{z}\right) = -k - k - k = -3k$

The electric field $\vec{E}$ at a point $\vec{r}$ inside the sphere is $\vec{E} = \frac{\rho_b r}{3\varepsilon_0}\hat{r} = \frac{-kr}{\varepsilon_0}\hat{r}$.

The electric field $\vec{E}$ at a point $\vec{r}$ outside the sphere is zero, since total bound charge is zero.

Ans. 32: (a)

Solution: $$\vec{E}=\begin{cases}0 & ; \quad r<R_1\\ \dfrac{Q}{4\pi\varepsilon r^2}\hat{r} & ; \quad R_1<r<R_2\\ \dfrac{Q}{4\pi\varepsilon_0 r^2}\hat{r} & ; \quad r>R_2\end{cases}$$

$$W=\frac{1}{2}\int_{all\ space}\vec{D}.\vec{E}d\tau=\frac{1}{2}\frac{Q^2}{(4\pi)^2}\left\{\frac{1}{\varepsilon}\int_{R_1}^{R_2}\frac{1}{r^4}4\pi r^2dr+\frac{1}{\varepsilon_0}\int_{R_2}^{\infty}\frac{1}{r^4}4\pi r^2dr\right\}$$

$$W=\frac{Q^2}{8\pi}\left\{\frac{1}{\varepsilon}\left(-\frac{1}{r}\right)\Big|_{R_1}^{R_2}+\frac{1}{\varepsilon_0}\left(-\frac{1}{r}\right)\Big|_{R_2}^{\infty}\right\}\Rightarrow W=\frac{Q^2}{8\pi\varepsilon_0\varepsilon_r}\left(\frac{1}{R_1}+\frac{\varepsilon_r-1}{R_2}\right)$$

NAT (Numerical Answer Type)

Ans. 33: 36

Solution: $\vec{E}=\dfrac{\lambda}{2\pi\varepsilon_0 r}\hat{r}=\dfrac{\lambda}{2\pi\varepsilon_0\sqrt{x^2+y^2}}\hat{r}=\dfrac{2\times9\times10^9\times20\times10^{-9}}{\sqrt{6^2+8^2}}=36\,\hat{r}\ V/m$

Ans. 34: 2

Solution: $E_o=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r_o^2}$ and $E_i=\dfrac{1}{4\pi\varepsilon_0}\dfrac{qr_i}{R^3}\Rightarrow\dfrac{E_i}{E_o}=\dfrac{r_ir_o^2}{R^3}\Rightarrow\dfrac{E_i}{E}=\dfrac{5(20)^2}{(10)^3}\Rightarrow E_i=2E$

Ans. 35: 0.6

Solution: $\vec{E}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{qr}{R^3}\hat{r};\ r<R\ and\quad \vec{E}=\dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}\hat{r};\ r>R$

$$W=\frac{\varepsilon_0}{2}\frac{q^2}{(4\pi\varepsilon_0)^2}\left\{\int_R^{\infty}\frac{1}{r^4}(r^2\,4\pi dr)+\int_0^R\left(\frac{r}{R^3}\right)^2(4\pi r^2dr)\right\}=\frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{R}+\frac{1}{5R}\right\},$$

$$W=\frac{1}{4\pi\varepsilon_0}\frac{3q^2}{5R}$$

Ans. 36: 8

Solution: Electrostatic potential energy $U \propto \frac{1}{r^3} \Rightarrow \frac{U_1}{U_2} = \frac{r_2^3}{r_1^3} = 8$

Ans. 37: **1**

Solution: Inside the square, there is only one point where field vanishes.

Ans. 38: **10**

Solution: $\vec{E} = -\vec{\nabla}V = -2\hat{x} - 4\hat{y} \Rightarrow \left|\vec{E}\right| = 20\,V/m$

Electrostatic energy density $= \frac{1}{2}\varepsilon_0 \left|\vec{E}\right|^2 = \frac{1}{2}\varepsilon_0 \times 20 = 10\varepsilon_0\, J/m^3$

Ans. 39: **1.41**

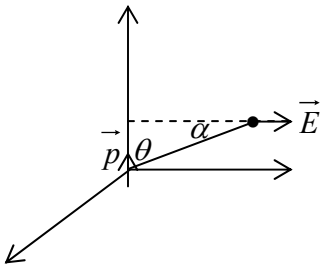
Solution: $\sigma_b = \sigma_d \Rightarrow \frac{Q}{4\pi b^2} = \frac{2Q}{4\pi d^2} \Rightarrow d = \sqrt{2}b$

Ans. 40: 1.41

$$\vec{E}(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

$$\tan\alpha = \frac{E_\theta}{E_r} = \frac{1}{2}\tan\theta \qquad \because \alpha = 90 - \theta \Rightarrow \cot\theta = \frac{1}{2}\tan\theta$$

$$\Rightarrow \tan^2\theta = 2 \Rightarrow \theta = \tan^{-1}\sqrt{2}$$



Ans. 41: **0.037**

Solution: According to multipole expansion $Q_{mono} = -\frac{q}{2} + q - \frac{q}{2} + q - q = 0$

$$\vec{p} = q(a\hat{x} + a\hat{y}) - \frac{q}{2}(a\hat{x} + a\hat{y}) - q(a\hat{x} - a\hat{y}) + q(-a\hat{x} - a\hat{y}) - \frac{q}{2}(-a\hat{x} + a\hat{y}) + 0 = 0$$

Thus $V \propto \frac{1}{r^3} \Rightarrow \frac{V(3r)}{V(r)} = \frac{1}{27} = 0.037$

MSQ (Multiple Select Questions)

Ans. 42: (a), (b) and (c)

Solution: Check: $\vec{E} = -\vec{\nabla}V$

Ans. 43: (a), (b) and (d)

Solution: $\vec{\nabla}\times\vec{E}_1 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & 0 \end{vmatrix} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(0-0) = 0$

$$\vec{\nabla}\times\vec{E}_2 = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & y^2 & 0 \end{vmatrix} = \hat{x}(0-0) - \hat{y}(0-0) + \hat{z}(0-2xy) = -2xy\hat{z}$$

Ans. 44: (b), (d)

Solution: $\vec{E} = -\vec{\nabla}V = -2\phi_0\left(x\hat{x} + y\hat{y} + z\hat{z}\right)$ and $\nabla^2\phi = -\frac{\rho}{\varepsilon_0} \Rightarrow \rho = -6\phi_0\varepsilon_0$

Ans. 45: (a), (b), (c) and (d)

Ans. 46: (a) and (c)

$$\vec{E} = \frac{q}{4\pi\varepsilon r^2}\hat{r} = \frac{q}{4\pi\varepsilon_0(1+\chi_e)}\frac{\hat{r}}{r^2} \Rightarrow \vec{P} = \varepsilon_0\chi_e\vec{E} = \frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{r}}{r^2}$$

$$\Rightarrow \sigma_b = \vec{P}.\hat{n} = \frac{q\chi_e}{4\pi(1+\chi_e)R^2} \Rightarrow Q_{\sigma_b} = \sigma_b \times 4\pi R^2 = \frac{q\chi_e}{1+\chi_e}$$

$$\Rightarrow Q_{\rho_b} = -Q_{\sigma_b} = -\frac{q\chi_e}{1+\chi_e}$$

Ans. 47: (b) and (c)

Solution: $\rho_b = -\vec{\nabla}.\vec{P} = -3k$

The electric field $\vec{E}$ at a point $\vec{r}$ inside the sphere is $\vec{E} = \frac{\rho_b r}{3\varepsilon_0}\hat{r} = \frac{-kr}{\varepsilon_0}\hat{r}$.

The electric field $\vec{E}$ at a point $\vec{r}$ outside the sphere is zero, since total bound charge is zero.

Ans. 48: (a), (b) and (c)

Solution: $\vec{E}=\frac{\rho r}{3\varepsilon}\hat{r}$ for $r<R$ and $\vec{E}=\frac{\rho R^3}{3\varepsilon_0 r^2}\hat{r}$ for $r>R$

$$V(0)=-\int_{\infty}^{0}\vec{E}.d\vec{l}=-\int_{\infty}^{R}\frac{\rho R^3}{3\varepsilon_0 r^2}dr-\int_{R}^{0}\frac{\rho r}{3\varepsilon}dr\Rightarrow V(0)=\frac{\rho R^3}{3\varepsilon_0}+\frac{\rho}{3\varepsilon}\frac{R^2}{2}=\frac{\rho R^2}{3\varepsilon_0}\left(1+\frac{1}{2\varepsilon_r}\right)$$

$$\Rightarrow V(0)=\frac{Q}{4\pi\varepsilon_0 R}\left(1+\frac{1}{2\varepsilon_r}\right),\text{ since }\rho=\frac{Q}{\frac{4}{3}\pi R^3}$$

Ans. 49: (a) and (d)

Solution: $\because E_1^{II}=E_2^{II}\Rightarrow E_2^{II}=3\hat{i}-5\hat{j}$

$$\text{and}\,\sigma_f=0\Rightarrow D_1^{\perp}=D_2^{\perp}\Rightarrow E_2^{\perp}=\frac{\varepsilon_1}{\varepsilon_2}E_1^{\perp}=\frac{5}{4}\left(+4\hat{k}\right)=5\hat{k}$$

$$\Rightarrow\vec{E}_2=\left(3\hat{i}-5\hat{j}+5\hat{k}\right)kV/m\Rightarrow\vec{D}_2=\left(12\hat{i}-20\hat{j}+20\hat{k}\right)kV/m$$

Chapter - 2

Magnetostatics

The magnetic field at any point due to steady current is called as magnetostatic field.

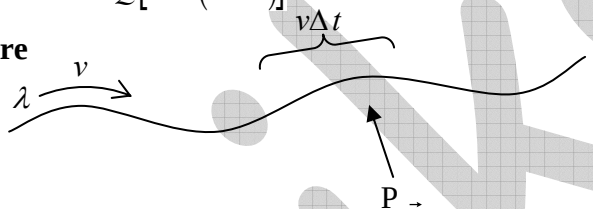
2.1 Magnetic Force on Current Element

The magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is, $\vec{F}_{mag} = Q(\vec{v} \times \vec{B})$. This is known as Lorentz force law.

In the presence of both electric and magnetic fields, the net force on Q would be:

$$\vec{F} = Q\left[\vec{E} + (\vec{v} \times \vec{B})\right]$$

2.1.1 Current in a Wire



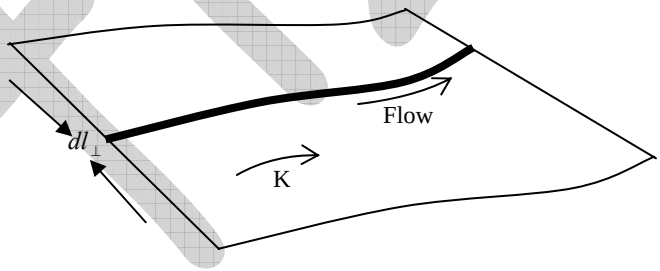
A line charge λ traveling down a wire at a speed $\vec{v}$ constitutes a current $\vec{I} = \lambda\vec{v}$.

Magnetic force on a segment of current-carrying wire is,

$$\vec{F}_{mag} = \int (\vec{v} \times \vec{B})dq = \int (\vec{v} \times \vec{B})\lambda dl = \int (\vec{I} \times \vec{B})dl .$$

Since $\vec{I}$ and $d\vec{l}$ points in the same direction $\vec{F}_{mag} = \int I\left(d\vec{l} \times \vec{B}\right) = I\int \left(d\vec{l} \times \vec{B}\right)$

2.1.2 Surface Current Density



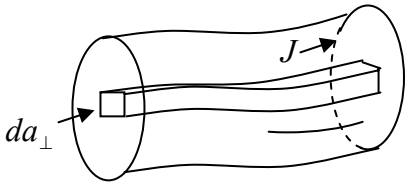
When charge flows over a surface, we describe it by the surface current $\vec{K}$.

$\vec{K} = \dfrac{d\vec{\mathrm{I}}}{dl_{\perp}}$ is the current per unit width-perpendicular to flow.

Also $\vec{K} = \sigma\vec{v}$ where is σ surface charge density and $\vec{v}$ is its velocity.

Magnetic force on surface current $\vec{F}_{mag} = \int (\vec{v} \times \vec{B})\sigma da = \int (\vec{K} \times \vec{B})da$

2.1.3 Volume Current Density



Flow

When the flow of charge is distributed throughout a three-dimensional region, we describe it by the volume current density $\vec{J}$.

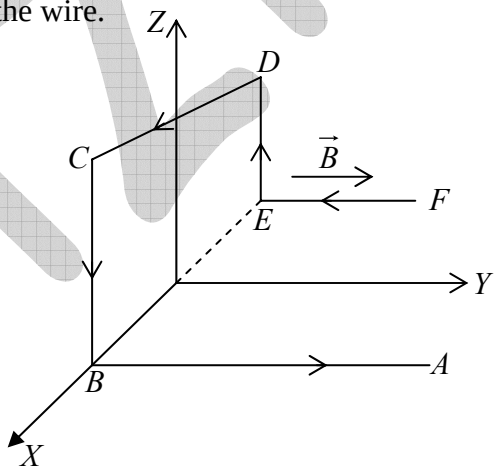
$\vec{J} = \frac{d\vec{I}}{da_\perp}$ is the current per unit area-perpendicular to flow.

Also $\vec{J} = \rho\vec{v}$ where ρ is volume charge density and $\vec{v}$ is its velocity.

Magnetic force on volume current $\vec{F}_{mag} = \int\left(\vec{v}\times\vec{B}\right)\rho d\tau = \int\left(\vec{J}\times\vec{B}\right)d\tau$

Current crossing a surface S is $I = \int_S \vec{J}\cdot d\vec{a}$

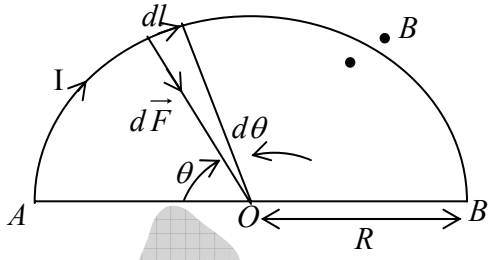
Example: A wire $ABCDEF$ (with each of side of length L) bent as shown in figure and carrying a current I is placed in a uniform magnetic induction B parallel to the positive y-direction. Find the force experienced by the wire.



Solution: FE and BA are parallel to magnetic induction $\vec{B}$. Magnetic force on each of them will be zero. DE and CB are perpendicular to $\vec{B}$. They carry currents in opposite directions forces on them will be equal in magnitude and opposite in direction. The net force due to these portions of wire will be zero. Now force on side DC is $\vec{F}_{DC} = ILB\hat{z}$.

Example: A semi–circular wire of radius R carries a current I and is placed in a uniform field B acting perpendicular to the plane of the semi–circle. Calculate force acting on the wire.

Solution: Consider an element of length dl of the wire.



The $d\vec{F}$ force on this element is obtained by

$$d\vec{F} = I(d\vec{l} \times \vec{B}) = Idl B \sin 90^0$$

$$\Rightarrow dF = I(Rd\theta)B \quad \text{since } dl = Rd\theta$$

$$F = \int_0^{\pi} dF \sin\theta = 2IBR$$

(Horizontal component cancels only perpendicular component add up).

2.2 Continuity Equation

The total charge per unit time leaving a volume V is $\oint \vec{J}.d\vec{a} = \int_V \left(\vec{\nabla}.\vec{J}\right) d\tau$.

Because charge is conserved, whatever flows out through the surface must come at the expense of that remaining inside:

$$\int_V \left(\vec{\nabla}.\vec{J}\right) d\tau = -\frac{d}{dt}\int_V \rho d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau .$$

(The minus sign reflects the fact that an outward flow decreases the charge left in V.) Since this applies to any volume, we conclude that

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} .$$

This is the precise mathematical statements of **local charge conservation**.

Note: When a steady current flows in a wire, its magnitude I must be the same all along the line; otherwise, charge would be piling up somewhere, and it wouldn't be a steady current.

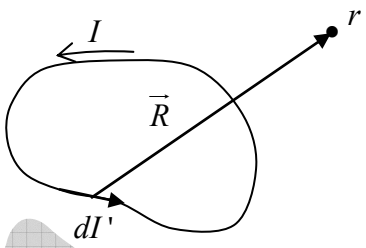
Thus for magnetostatic fields $\frac{\partial \rho}{\partial t} = 0$ and hence the continuity equation becomes:

$$\vec{\nabla} \cdot \vec{J} = 0 .$$

2.3 Biot-Savart Law

The magnetic field of a steady line current is given by

$$\vec{B}(r)=\frac{\mu_0}{4\pi}\int\frac{\vec{I}\times\hat{R}}{R^2}dl'=\frac{\mu_0}{4\pi}I\int\frac{d\vec{l}'\times\hat{R}}{R^2}$$



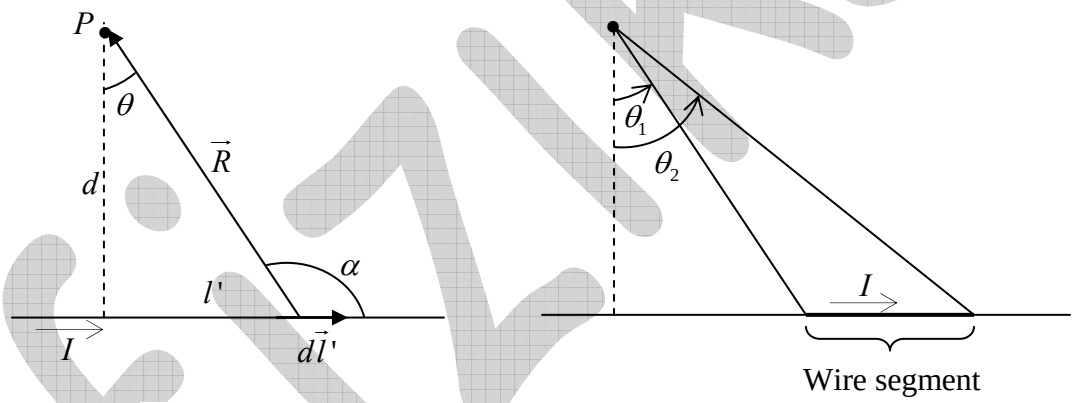
$$where\ \mu_0=4\pi\times10^{-7}\frac{N}{A^2}(permeability\ of\ free\ space)$$

For surface and volume current Biot-Savart law becomes:

$$\mathrm{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{K}(r')\times\hat{R}}{R^2}da'\ and\ \mathrm{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')\times\hat{R}}{R^2}d\tau'.$$

2.3.1 Magnetic Field due to Wire

Let us find the magnetic field a distance *d* from a long straight wire carrying a steady current *I*.



In the diagram, $(d\vec{l}'\times\hat{R})$ points out of the page and has magnitude $dl'\sin\alpha = dl'\cos\theta$

Since $l'=d\tan\theta\Rightarrow dl'=\frac{d}{\cos^2\theta}d\theta\Rightarrow dl'=\frac{d}{\cos^2\theta}d\theta$ and $d=R\cos\theta\Rightarrow\frac{1}{R^2}=\frac{\cos^2\theta}{d^2}$

From Biot–Savart law: $\vec{B}(\vec{r})=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}'\times\hat{R}}{R^2}=\frac{\mu_0 I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{\cos^2\theta}{d^2}\times\frac{d}{\cos^2\theta}\times\cos\theta\,d\theta$

$$B=\frac{\mu_0 I}{4\pi d}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta=\frac{\mu_0 I}{4\pi d}(\sin\theta_2-\sin\theta_1)$$

For Infinite wire: $\theta_1=-\frac{\pi}{2}$ and $\theta_2=\frac{\pi}{2}\Rightarrow\vec{B}=\frac{\mu_0 I}{2\pi d}\hat{\phi}$

Note:

1. Magnetic field a distance r from a long straight wire carrying a steady current I is

$$B = \frac{\mu_0 I}{4\pi r}\left(\sin\theta_2 - \sin\theta_1\right).$$

2. Magnetic field a distance r from a infinite wire carrying a steady current I is:

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}.$$

3. Force (per unit length) of attraction between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 in same direction are:

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}.$$

4. If currents are in opposite direction they will repel with same magnitude.

2.3.2 Magnetic Field due to Solenoid and Toroid

The magnetic field of a very long solenoid, consisting of n closely wound turns per unit length of a cylinder of radius R and carrying a steady current I is:

$$\vec{B} = \begin{cases} \mu_0 nI\hat{z} & \text{inside the solenoid} \\ 0 & \text{outside the solenoid.} \end{cases}$$

Magnetic field due to Toroid is

$$\vec{B} = \begin{cases} +\dfrac{\mu_0 NI}{2\pi r}\hat{\phi} & \text{for points inside the coil} \\ 0 & \text{for points outside the coil} \end{cases}$$

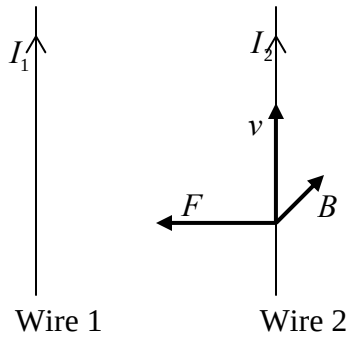
where N is the total number of turns.

Example: Find the force of attraction between two long, parallel wires a distance d apart, carrying current I_1 and I_2 in the same direction.

Solution: The field at (2) due to (1) is $B = \dfrac{\mu_0 I_1}{2\pi d}$ (points into the page).

Force on (2) is $F = I_2 \times \dfrac{\mu_0 I_1}{2\pi d}\int dl \Rightarrow f = \dfrac{\mu_0 I_1 I_2}{2\pi d}$

Force per unit length is towards (1) and net force is attractive



Example: Find the magnetic field a distance d above the center of a circular loop of radius R, which carries a steady current I.

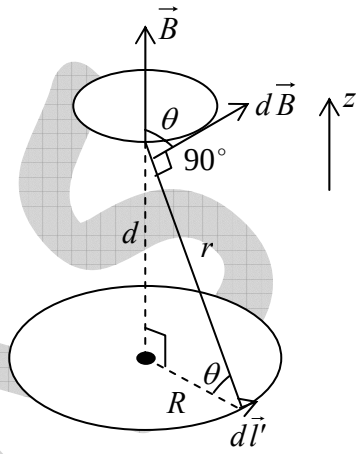
Solution: The field $d\vec{B}$ attributable to the segment $d\vec{l}\,'$ as shown. As we integrate $d\vec{l}\,'$ around the loop, $d\vec{B}$ sweeps out a cone. The horizontal components cancel, and the vertical components combine to give.

$$B(z)=\frac{\mu_0 I}{4\pi}\int\frac{dl'}{r^2}\cos\theta;\quad d\vec{l}\,'\text{and }\hat{r}\ \ are\perp^{r}$$

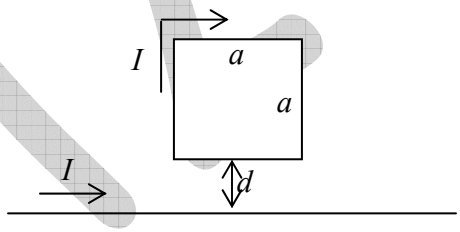
$$B(z)=\frac{\mu_0 I}{4\pi}\frac{\cos\theta}{r^2}\int d\vec{l}\,'=\frac{\mu_0 I}{4\pi}\left(\frac{\cos\theta}{r^2}\right)\times 2\pi R,$$

Thus $\vec{B}=\frac{\mu_0 I}{2}\frac{R^2}{(R^2+d^2)^{3/2}}\hat{z}$

At the center of the circle $\vec{B}(0)=\frac{\mu_0 I}{2R}\hat{z}$



Example: Find the force on a square loop placed as shown in figure, near an infinite straight wire. Both the loop and the wire carry a steady current I.



Solution: The force on the two sides cancels.

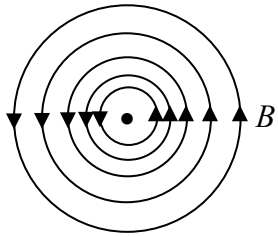
At the bottom, $B=\frac{\mu_0 I}{2\pi d}\Rightarrow F=\left(\frac{\mu_0 I}{2\pi d}\right)Ia=\frac{\mu_0 I^2 a}{2\pi d}(up)$

At the top, $B=\frac{\mu_0 I}{2\pi(d+a)}\Rightarrow F=\frac{\mu_0 I^2 a}{2\pi(d+a)}(\text{down})$

Thus Net Force $=\frac{\mu_0 I^2 a^2}{2\pi d(d+a)}(\text{up})$

2.4 Ampere's Law

The magnetic field of an infinite wire is shown in the figure (the current is coming out of the page). Let us find the integral of $\vec{B}$ around a circular path of radius *r*, centered at the wire, is



$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_o I}{2\pi r} dl = \mu_o I\,.$$

Notice that the answer is independent of *r*; that is because *B* decreases at the same rate as the circumference increases. If we use cylindrical coordinates (r,ϕ,z), with the current flowing along the *z* axis, $\vec{B} = \frac{\mu_o I}{2\pi r}\hat{\phi}$

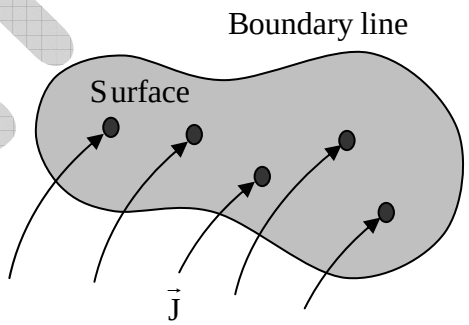
In general we can write $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$

where I_{enc} is the total current enclosed by the amperian loop.

Since $\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow \int (\vec{\nabla}\times\vec{B}).d\vec{a} = \mu_0 \int \vec{J}.d\vec{a} \;\Rightarrow \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

Right hand Rule

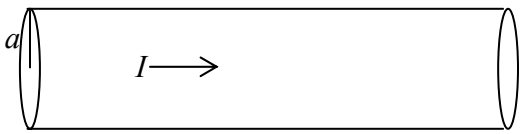
If the fingers of your right hand indicate the direction of integration around the boundary, then your thumb defines the direction of a positive current.



Example: A steady current *I* flow down a long cylindrical wire of radius *a*. Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that *J* is proportion to *r*, the distance from the axis.



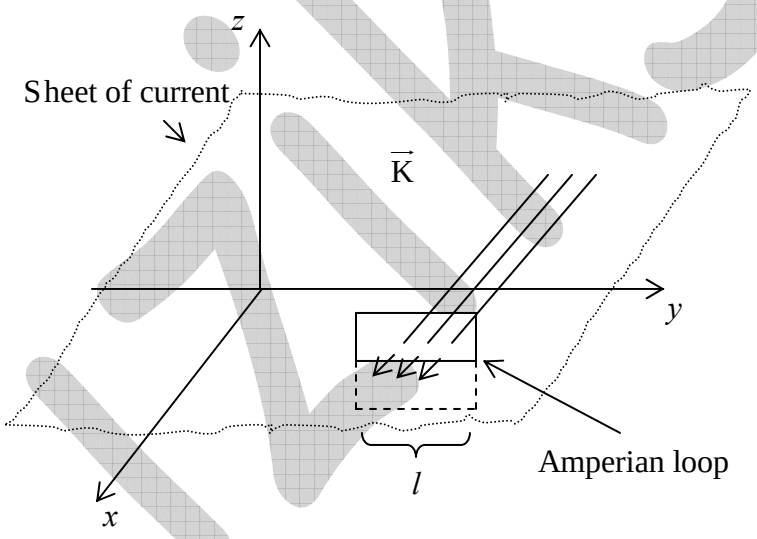
Solution: (a) $\oint \vec{B}.d\vec{l} = B2\pi r = \mu_0 I_{enc} \Rightarrow \vec{B} = \begin{cases} 0 & \text{for r} < a \\ \frac{\mu_0 I}{2\pi r}\hat{z} & \text{for r} > a \end{cases}$

(b) $J = kr$, $I = \int_0^a J da = \int_0^a kr(2\pi r dr) = \frac{2\pi k a^3}{3} \Rightarrow k = \frac{3I}{2\pi a^3}$

$$\left.\begin{aligned} I_{enc} &= \int_0^r J da = \int_0^r kr' \times 2\pi r' dr' = \frac{2\pi k r^3}{3} = \frac{I r^3}{a^3} \quad for\ r < a \\ I_{enc} &= I \quad for\ r > a \end{aligned}\right\} \Rightarrow \vec{B} = \begin{cases} \frac{\mu_0 I r^2}{2\pi a^3}\hat{\phi} & for\ r < a \\ \frac{\mu_0 I}{2\pi r}\hat{\phi} & for\ r > a \end{cases}$$

Example: Find the magnetic field of an infinite uniform surface current $\vec{K} = K\hat{x}$, flowing over the *x*–*y* plane.

Solution:



Since $\vec{K} = K\hat{x} \Rightarrow \vec{B}$ have no *x*-component because B is $\perp^r$ to *x*-component i.e. in the direction of $(\vec{K} \times \hat{R})$.

Also, $\vec{B}$ have no *z*-component: For $y > 0$, B is along $\hat{z}$ and for $y < 0$, B is along $-\hat{z}$, thus field cancels each other.

$\Rightarrow \vec{B}$ **has only *y*-component:**

For $z > 0$, $\vec{B}$ points left $(-\hat{y})$ and for $z < 0$, $\vec{B}$ point's right $(\hat{y})$.

Draw a rectangular amperian loop parallel to the *yz* plane and extending an equal distance above and below the surface. Now apply ampere's law, we find

$\oint \vec{B}.d\vec{l} = 2Bl = \mu_0 I_{enc} = \mu_0 Kl$ {One Bl from top segment, and the other from bottom}

$$B = \frac{\mu_0 K}{2} \Rightarrow \vec{B} = \begin{cases} \frac{-\mu_0 K}{2}\hat{y} & for\ z > 0 \\ \frac{\mu_0 K}{2}\hat{y} & for\ z < 0 \end{cases}$$

Note: The field is independent of the distance from the plane, just like the electric field of a uniform surface charge.

2.5 Magnetic Vector Potential $\left(\vec{A}\right)$

Since $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

For magnetostatic fields, $\vec{\nabla}\cdot\vec{A} = 0$ and $\nabla^2\vec{A} = -\mu_0\vec{J}$

If $\vec{J}$ goes to zero at infinity, $\vec{A}(r) = \frac{\mu_0}{4\pi}\int\frac{\vec{J}(r')}{R}d\tau'$ for volume current.

For line and surface currents, $\vec{A}(r) = \frac{\mu_0 I}{4\pi}\int\frac{1}{R}dl'$; $\vec{A}(r) = \frac{\mu_0}{4\pi}\int\frac{\vec{K}}{R}da'$

Example: What current density would produce the vector potential $\vec{A} = K\hat{\phi}$ (where K is a constant), in cylindrical coordinates ?

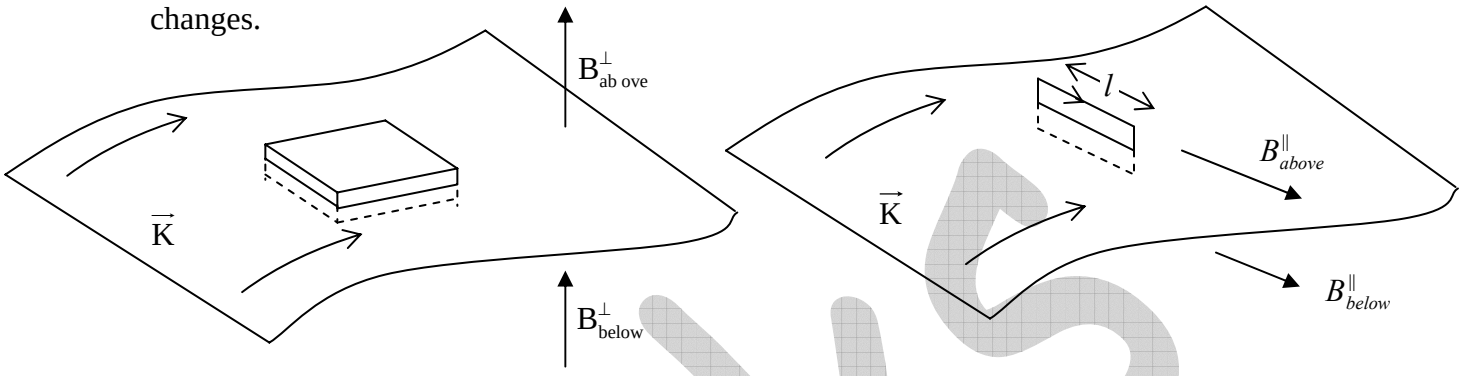
Solution: $A_\phi = K \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\hat{z} = \frac{K}{r}\hat{z}$

$$\left\{\vec{\nabla}\times\vec{A} = \left[\frac{1}{r}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\hat{r} + \left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right]\hat{\phi} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial_\phi}\right]\hat{z}\right\}$$

Since $\vec{J} = \frac{1}{\mu_0}(\vec{\nabla}\times\vec{B}) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial r}\left(\frac{K}{r}\right)\right]\hat{\phi} = \frac{K}{\mu_0 r^2}\hat{\phi} \Rightarrow \vec{J} = \frac{K}{\mu_0 r^2}\hat{\phi}$

2.6 Magnetostatic Boundary Condition (Boundary is sheet of current, $\vec{K}$)

Just as the electric field suffers a discontinuity at a surface charge, so the magnetic field is discontinuous at a surface current. Only this time it is the tangential component that changes.



Since $\oint \vec{B}.d\vec{a} = 0 \Rightarrow B^{\perp}_{above} = B^{\perp}_{below}$

For tangential components

$$\begin{cases} \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} \Rightarrow B^{\parallel}_{above} - B^{\parallel}_{below} = \mu_0 K & \left(\vec{B} \text{ is parallel to surface but } \perp^r \text{ to } \vec{K}\right) \\ \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} \Rightarrow B^{\parallel}_{above} = B^{\parallel}_{below} & \left(\vec{B} \text{ is parallel to surface and along } \vec{K}\right) \end{cases}$$

Thus the component of $\left(\vec{B}\right)$ that is parallel to the surface but perpendicular to the current is discontinuous in the amount $\mu_0 K$. A similar amperian loop running parallel to the current reveals that the parallel component is continuous. The result can be summarized in a single formula: $\vec{B}_{above} - \vec{B}_{below} = \mu_0\left(\vec{K} \times \hat{n}\right)$

where $\hat{n}$ is a unit vector perpendicular to the surface, pointing "upward".

Like the scalar potential in electrostatics, the vector potential is continuous across, a boundary: $\vec{A}_{above} = \vec{A}_{below}$

For $\vec{\nabla}.\vec{A} = 0$ guarantees that the normal component is continuous, and $\vec{\nabla} \times \vec{A} = \vec{B}$, in the form $\oint_{line} \vec{A}.d\vec{l} = \int_S \vec{B}.d\vec{a} = \phi$

But the derivative of $\left(\vec{A}\right)$ inherits the discontinuity of $\left(\vec{B}\right)$:

$$\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}\ .$$

Example: (a) Find the magnetic vector potential at a distance r from an infinite straight wire carrying a current I.

(b) Find the magnetic potential inside the wire, if it has radius R and the current is uniformly distributed.

Solution: (a) $\vec{A}$ point in the same direction as I and is a function of r (the distance from the wire). In cylindrical coordinates $\vec{A} = A(r)\hat{z}$ and $\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A}{\partial r}\hat{\phi} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$.

$\Rightarrow \frac{\partial A}{\partial r} = -\frac{\mu_0 I}{2\pi r} \Rightarrow \vec{A}(r) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{r}{a}\right)\hat{z}$ (Constant a is arbitrary)

Verify that $\vec{\nabla} \cdot \vec{A} = 0$ and $\vec{\nabla} \times \vec{A} = \vec{B}$

(b) $\oint \vec{B}.d\vec{l} = B.2\pi r = \mu_0 I_{enc} = \mu_0 J\pi r^2 = \frac{\mu_0 I}{\pi R^2}\pi r^2 = \frac{\mu_0 I r^2}{R^2} \Rightarrow \vec{B} = \frac{\mu_0}{2\pi}\frac{Ir}{R^2}\hat{\phi}$

$\vec{B} = \vec{\nabla} \times \vec{A} = -\frac{\partial A_z}{\partial r}\hat{\phi} \Rightarrow \frac{\partial A}{\partial r} = -\frac{\mu_0 I}{2\pi}\frac{Ir}{R^2} \Rightarrow \vec{A} = -\frac{\mu_0 I}{4\pi R^2}(r^2 - b^2)\hat{z}$

where b is arbitrary constant.

$\vec{A}$ must be continuous at R, $-\frac{\mu_0 I}{2\pi}\ln\left(\frac{R}{a}\right) = -\frac{\mu_0 I}{4\pi R^2}(R^2 - b^2)$, which means that we must pick a and b such that $2\ln\left(\frac{R}{a}\right) = 1 - \frac{b^2}{R^2}$.

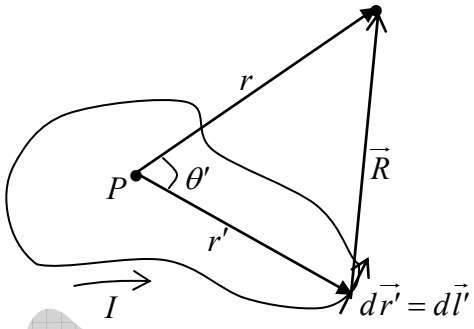
Example: Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Solution: Since $\oint_{line} \vec{A}.d\vec{l} = \int_S \left(\vec{\nabla} \times \vec{A}\right).d\vec{a} = \int_S \vec{B}.d\vec{a} = \phi$, where ϕ is the flux of $\vec{B}$ through the loop in question.

Inside solenoid: $\oint_{line} \vec{A}.d\vec{l} = A(2\pi r) = \int_S \vec{B}.d\vec{a} = \mu_0 nI\left(\pi r^2\right) \Rightarrow \vec{A} = \frac{\mu_0 nI}{2} r\hat{\phi}, \quad \text{for } r < R.$

Outside solenoid: $A(2\pi r) = \int_S \vec{B}.d\vec{a} = \mu_0 nI\left(\pi R^2\right) \Rightarrow \vec{A} = \frac{\mu_0 nI}{2}\frac{R^2}{r}\hat{\phi}, \quad \text{for } r > R.$

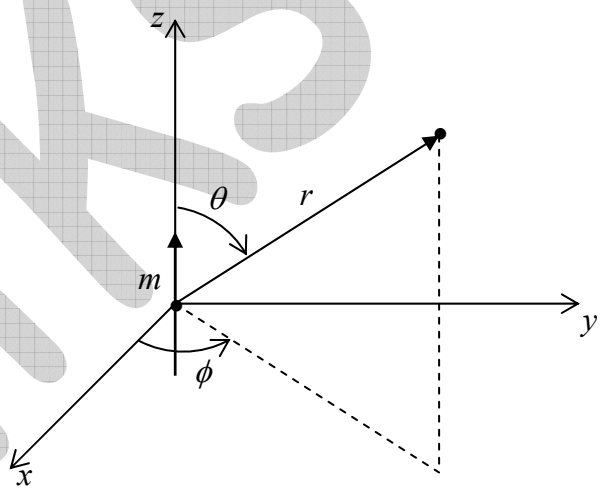
2.7 Multiple Expansion of Vector Potential



We can derive approximate formula for the vector potential of a localized current distribution, valid at distant points. We can always write the potential in the form of a power series in $\frac{1}{r}$, where r is the distance to the point in question. Thus we can always write

$$\vec{A}(r)=\frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}'+\frac{1}{r^2}\oint r'\cos\theta'\,d\vec{l}'+\frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta'-\frac{1}{2}\right)d\vec{l}'+.....\right]$$

First term, *monopole* $\Rightarrow \oint d\vec{l}'=0$ (no magnetic monopole)



Second term, *dipole* $\vec{A}_{dip}(r)=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$

where $\vec{m}$ is the magnetic dipole moment: $\vec{m}=I\int d\vec{a}=I\vec{A}$ where $\vec{A}$ is area vector

Thus $\vec{A}_{dip}\left(\vec{r}\right)=\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$

Hence

$$\vec{B}_{dip}\left(\vec{r}\right)=\vec{\nabla}\times\vec{A}=\frac{\mu_0 m}{4\pi\, r^3}\left(2\cos\theta\,\hat{r}+\sin\theta\hat{\theta}\right)=\frac{\mu_0}{4\pi}\cdot\frac{1}{r^3}\left[3\left(\vec{m}\cdot\hat{r}\right)\hat{r}-\vec{m}\right]$$

Note: (a) When a magnetic dipole is placed in a uniform magnetic field ($\vec{B}$), net force on the dipole is zero and it experiences a torque $\vec{\tau}=\vec{m}\times\vec{B}$.

(b) In non-uniform field, dipoles have net force $\vec{F}=\vec{\nabla}\left(\vec{m}\cdot\vec{B}\right)$ and torque $\vec{\tau}=\vec{m}\times\vec{B}$.

(c) Energy of an ideal dipole $\vec{m}$ in an magnetic field $\vec{B}$ is $U=-\vec{m}.\vec{B}$.

(d) Interaction energy of two dipoles separated by a distance $\vec{r}$ is

$$U=\frac{1}{4\pi\varepsilon_0 r^3}\left[\vec{m_1}\cdot\vec{m_2}-3\left(\vec{m_1}\cdot\hat{r}\right)\left(\vec{m_2}\cdot\hat{r}\right)\right]$$

Example: A phonograph record of radius *R*, carrying a uniform surface charge σ is rotating at constant angular velocity ω. Find its magnetic dipole moment.

Solution: Magnetic moment of a ring of radius *r* and thickness dr is, $dm = I\pi r^2$ where

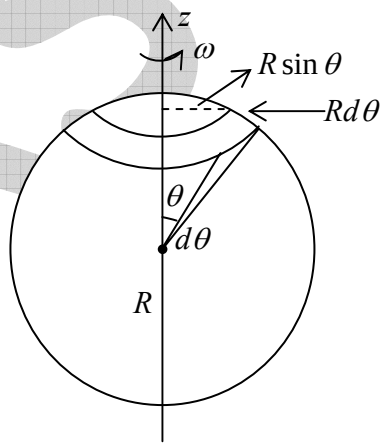
$$I = \sigma v dr = \sigma r \omega dr$$

$$\Rightarrow m = \int_0^R \pi r^2 \times \sigma \omega r dr = \pi\sigma\omega \int_0^R r^3 dr = \frac{\pi\sigma\omega R^4}{4} \Rightarrow \vec{m} = \frac{\pi\sigma\omega R^4}{4}\hat{z}$$

Example: A spherical shell of radius *R*, carrying a uniform surface charge σ, is set spinning at angular velocity ω. Find its Magnetic dipole moment.

Solution: The total charge on the shaded ring is

$$dq = \sigma(2\pi R\sin\theta)Rd\theta$$



Time for one revolution is $dt = \frac{2\pi}{\omega}$

$\Rightarrow$Current in the ring $I = \frac{dq}{dt} = \sigma\omega R^2 \sin\theta d\theta$

Area of the ring $= \pi(\mathrm{R}\sin\theta)^2$, so the magnetic moment of the ring is

$$dm = (\sigma\omega R^2 \sin\theta d\theta) \times \pi R^2 \sin^2\theta$$

$$m = \sigma\omega R^4 \int_0^\pi \sin^3\theta d\theta = \frac{4}{3}\pi \times \sigma\omega R^4 \Rightarrow \vec{m} = \frac{4\pi}{3}\sigma\omega R^4 \hat{z}$$

2.8 Magnetisation $\left(\vec{M}\right)$

If a piece of magnetic material is examined on an atomic scale we will find *tiny currents*: electrons orbiting around nuclei and electrons spinning about their axes.

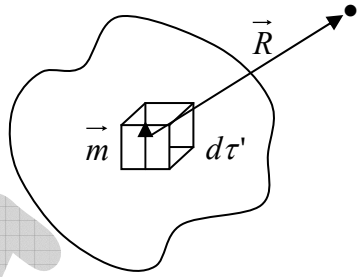
For macroscopic purpose, these current loops are so small that we may treat them as magnetic dipoles. Ordinarily they cancel each other out because of the random orientation of the atoms. But when a magnetic field is applied, a net alignment of these magnetic dipoles occurs, and medium becomes magnetically polarized, or **magnetized**.

Magnetization $\left(\vec{M}\right)$ is magnetic dipole moment per unit volume.

2.8.1 The Field of a Magnetized Object (Bound Currents)

Consider a piece of magnetized material with magnetization $\left(\vec{M}\right)$.

Then the vector potential of a single dipole $\left(\vec{m}\right)$ is given by



$$\vec{A}(r)=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{R}}{R^2}$$

In the magnetized object, each volume element $d\tau'$ carries a dipole moment $\vec{M}d\tau'$, so the total vector potential is $\vec{A}(r)=\frac{\mu_0}{4\pi}\int_v\frac{\vec{M}(r')\times\hat{R}}{R^2}d\tau'$

The equation can be written as $\vec{A}(r)=\frac{\mu_0}{4\pi}\int_v\frac{1}{R}\left[\vec{\nabla}'\times\vec{M}(r')\right]d\tau'+\frac{\mu_0}{4\pi}\oint\frac{1}{R}\left[\vec{M}(r')\times d\vec{a}'\right]$.

The first term is like potential of a volume current

$$\vec{J}_b=\vec{\nabla}\times\vec{M}$$

while the second term is like potential of a surface current,

$$\vec{K}_b=\vec{M}\times\hat{n}$$

where $\hat{n}$ is the normal to the unit vector.

With these definitions, the field of a magnetized object is

$$\vec{A}(r)=\frac{\mu_0}{4\pi}\int_v\frac{\vec{J}_b(r')}{R}d\tau'+\oint\frac{\vec{K}_b(r')}{R}da'.$$

This means the potential(and hence also the field) of a magnetized object is the same as would be produced by a volume current $\vec{J}_b=\vec{\nabla}\times\vec{M}$ throughout the material, plus a surface current $\vec{K}_b=\vec{M}\times\hat{n}$, on the boundary. We first determine these **bound currents**, and then find the field they produce.

Example: An infinitely long circular cylinder carries a uniform magnetization $\vec{M}$ parallel to its axis. Find the magnetic field (due to $\vec{M}$) inside and outside the cylinder.

Solution: $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0;\ \ \vec{K}_b = \vec{M}\times\hat{n} = M\hat{z}\times\hat{r} = M\hat{\phi}$

The field is that of a surface current $\vec{K}_b = M\hat{\phi}$, that is the case of a **solenoid**,

So the field outside is zero.

Field inside is: B=$\mu_0 K_b = \mu_0 M \Rightarrow \vec{B} = \mu_0\vec{M}$

Example: A long circular cylinder of radius *R* carries a magnetization $\vec{M} = Kr^2\hat{\phi}$ where *K* is a constant; *r* is the distance from the axis. Find the magnetic field due to $\vec{M}$, for points inside and outside the cylinder.

Solution: $\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{r}\frac{\partial}{\partial r}(rKr^2)\hat{z} = 3Kr\hat{z}$ *and* $\vec{K}_b = \vec{M}\times\hat{n} = KR^2(\hat{\phi}\times\hat{r}) = -KR^2\hat{z}$

So the bound current flows up the cylinder, and returns down the surface.

$I_{Total} = \int J_b da + \int K_b dl = \int_0^R (3kr)2\pi r dr + \left(-KR^2\times 2\pi R\right) = 2\pi KR^3 - 2\pi KR^3 = 0$.

Outside point: $I_{enc} = 0 \Rightarrow B = 0$

Inside point: $B.2\pi r = \mu_0 I_{enc} = \mu_0\int_0^r J_b da = 2\pi K\mu_0 r^3 \Rightarrow \vec{B} = \mu_0 Kr^2\hat{\phi} = \mu_0\vec{M}$

2.9 The Auxiliary field ($\vec{H}$)

2.9.1 Ampere's Law in in presence of Magnetic Materials

In a magnetized material the total current can be written as

$\vec{J} = \vec{J}_b + \vec{J}_f$ where $\vec{J}_b$ is bound current and $\vec{J}_f$ is free current.

$$\because \vec{\nabla}\times\vec{B} = \mu_0\vec{J} \Rightarrow \frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right) = \vec{J} = \vec{J}_b + \vec{J}_f = \left(\vec{\nabla}\times\vec{M}\right) + \vec{J}_f \Rightarrow \vec{\nabla}\times\left(\frac{\vec{B}}{\mu_0} - \vec{M}\right) = \vec{J}_f$$

The quantity in parentheses is designated by the letter $\vec{H}$ and $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

$\Rightarrow \vec{\nabla}\times\vec{H} = \vec{J}_f$

In integral form $\oint \vec{H}\cdot d\vec{l} = I_{f_{enc}}$ where $I_{f_{enc}}$ is the total free current passing through the amperian loop.

$\vec{H}$ plays a role in magnetostatic analogous to $\vec{D}$ in electrostatic: Just as $\vec{D}$ allowed us to write Gauss's law in terms of the free charge alone, $\vec{H}$ permits us to express Ampere's law in terms of the free current alone- and free current is what we control directly.

Note:

When we have to find $\vec{B}$ or $\vec{H}$ in a problem involving magnetic materials, first look for symmetry. If the problem exhibits cylindrical, plane, solenoid, or toroidal symmetry, then we can get $\vec{H}$ directly from the equation $\oint \vec{H} \cdot d\vec{l} = I_{f_{enc}}$.

Although this equation is valid for all the surfaces but it is useful for only symmetrical surfaces.

2.9.2 Magnetic Susceptibility and Permeability

For most substances magnetization is proportional to the field $\vec{H}$, $\Rightarrow \vec{M} = \chi_m \vec{H}$, where χ_m is magnetic susceptibility of the material.

$\Rightarrow \vec{B} = \mu_0\left(\vec{H} + \vec{M}\right) = \mu_0(1+\chi_m)\vec{H} \Rightarrow \vec{B} = \mu\vec{H}$ where $\mu = \mu_0\mu_r = \mu_0(1+\chi_m)$ is permeability of material.

2.9.3 Boundary Condition ($\vec{H}$)

The boundary between two medium is a thin sheet of free surface current K_f.

The Ampere's law states that

$$\oint_S \vec{H} d\vec{l} = I_{free}$$

$\Rightarrow \vec{H}_{above} - \vec{H}_{below} = \vec{K}_f \times \hat{n}$.

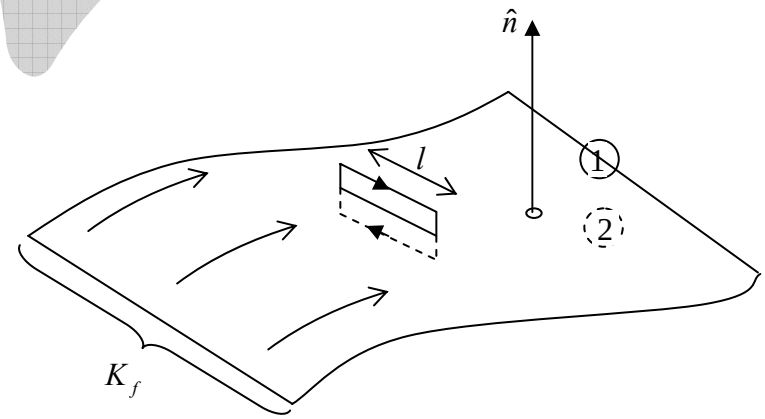
Since

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M} \quad \Rightarrow \vec{\nabla}.\vec{H} = -\vec{\nabla}.\vec{M}$$

since $\vec{\nabla}.\vec{B} = 0$

Thus

$$H^{\perp}_{above} - H^{\perp}_{below} = -\left(M^{\perp}_{above} - M^{\perp}_{below}\right)$$



Example: A current I flows down a long straight wire of radius a. If the wire is made of linear material with susceptibility χ_m, and the current is distributed uniformly, what is the magnetic field a distance r from the axis? Find all the bound currents. What is the net bound current following down the wire?

Solution: $\oint \vec{H}\cdot d\vec{l} = I_{f_{enc}} \Rightarrow H.2\pi r = \begin{cases} I\dfrac{r^2}{a^2} & ;r<a \\ I & ;r>a \end{cases} \Rightarrow \vec{H} = \begin{cases} \dfrac{Ir}{2\pi a^2}\hat{\phi} & ;r<a \\ \dfrac{I}{2\pi r}\hat{\phi} & ;r>a \end{cases}$

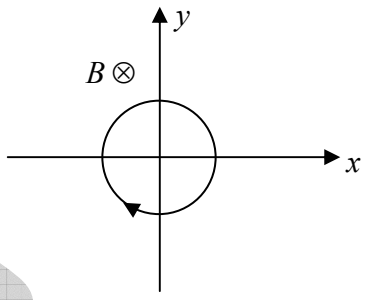
$\Rightarrow \vec{B} = \mu\vec{H} = \mu_0(1+\chi_m)\vec{H} \Rightarrow \vec{J}_b = \chi_m \vec{J}_f \Rightarrow J_b = \chi_m \dfrac{I}{\pi a^2}$ (same as I)

$\vec{K}_b = \vec{M}\times\hat{n} = \chi_m \vec{M}\times\hat{n} \Rightarrow K_b = \dfrac{\chi_m I}{2\pi a}$ (Opposite to I)

Total bound current $I_b = J_b \times \pi a^2 + K_b \times 2\pi a = \chi_m I - \chi_m I = 0$

MCQ (Multiple Choice Questions)

Q1. A conducting loop, carrying a current I, is placed in a uniform magnetic field pointing into the plane of the paper as shown. The loop will have a tendency to



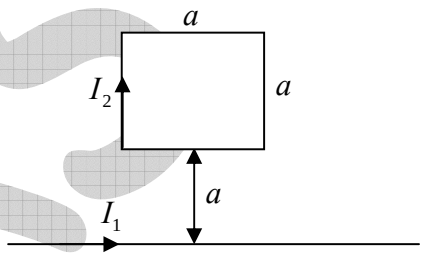
(a) contract

(b) expand

(c) move towards positive x-axis

(d) move towards negative x-axis

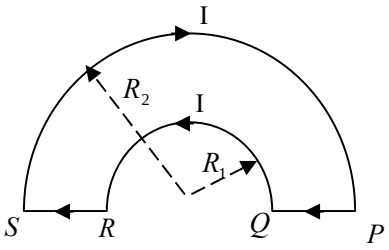
Q2. A square loop is placed near an infinite straight wire as shown in figure. The loop and wire carry a steady current I_2 and I_1 respectively. Then the force acting on the square loop is:



(a) $\frac{\mu_0 I_1 I_2}{2\pi a}$ (b) $\frac{\mu_0 I_1 I_2}{4\pi a}$

(c) $\frac{\mu_0 I_1 I_2}{2\pi}$ (d) $\frac{\mu_0 I_1 I_2}{4\pi}$

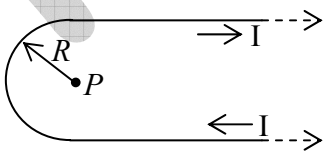
Q3. The wire loop $PQRSP$ formed by joining two semicircular wires of radii R_1 and R_2 carries a current I as shown in the figure. Then the magnetic field B at the centre is



(a) $\frac{\mu_0 I}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$, outward (b) $\frac{\mu_0 I}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$, inward

(c) $\frac{\mu_0 I}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$, outward (d) $\frac{\mu_0 I}{4}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$, inward.

Q4. The magnetic field at point P for the steady current configurations shown in figure is given by



(a) $B=\frac{\mu_0 I}{R}\left(1+\frac{2}{\pi}\right)$ (b) $B=\frac{\mu_0 I}{2R}\left(1+\frac{2}{\pi}\right)$

(c) $B=\frac{\mu_0 I}{3R}\left(1+\frac{2}{\pi}\right)$ (d) $B=\frac{\mu_0 I}{4R}\left(1+\frac{2}{\pi}\right)$

Q5. A uniform surface current is flowing in the positive *y*-direction over an infinite sheet lying in the *x*-*y* plane. The direction of the magnetic field is:

(a) along $-\hat{z}$ for $z>0$ and along $\hat{z}$ for $z<0$

(b) along $-\hat{x}$ for $z>0$ and along $\hat{x}$ for $z<0$

(c) along $\hat{z}$ for $z>0$ and along $\hat{x}$ for $z<0$

(d) along $\hat{x}$ for $z>0$ and along $-\hat{x}$ for $z<0$

Q6. Which one of the following current densities $\vec{J}$, can generate the magnetic vector potential $\vec{A}=\left(y^2\hat{x}+x^2\hat{y}\right)$?

(a) $\frac{2}{\mu_0}\left(x\hat{x}+y\hat{y}\right)$ (b) $-\frac{2}{\mu_0}\left(\hat{x}-\hat{y}\right)$ (c) $-\frac{2}{\mu_0}\left(\hat{x}+\hat{y}\right)$ (d) $\frac{2}{\mu_0}\left(x\hat{x}-y\hat{y}\right)$

Q7. Which of the following expressions for a vector potential $\vec{A}$ does not represent a uniform magnetic field of magnitude B_0 along the *z*-direction?

(a) $\vec{A}=(0,B_0x,0)$ (b) $\vec{A}=(-B_0y,0,0)$

(c) $\vec{A}=\left(\frac{B_0x}{2},\frac{B_0y}{2},0\right)$ (d) $\vec{A}=\left(-\frac{B_0y}{2},\frac{B_0x}{2},0\right)$

Q8. An infinitely long hollow cylinder of radius a carrying a surface charge density σ is rotated about its cylindrical axis with a constant angular speed ω. Then the magnitude of vector potential inside the cylinder at a distance r from its axis is:

(a) $2\mu_0\sigma a\omega r$ (b) $\mu_0\sigma a\omega r$ (c) $\frac{1}{2}\mu_0\sigma a\omega r$ (d) $\frac{1}{4}\mu_0\sigma a\omega r$

Q9. An infinitely long hollow cylinder of radius a carrying a surface charge density σ is rotated about its cylindrical axis with a constant angular speed ω. Then the magnitude of vector potential outside the cylinder at a distance r from its axis is:

(a) $2\mu_0\sigma a\omega r$ (b) $\frac{1}{2r}\mu_0\sigma a\omega R^2$

(c) $\frac{1}{2}\mu_0\sigma a\omega r$ (d) $\frac{1}{4r}\mu_0\sigma a\omega R^2$

Q10. The magnetic field corresponding to the vector potential $\vec{A} = \frac{1}{4}\vec{F} \times \vec{r} + \frac{10}{r^3}\vec{r}$ where $\vec{F}$ is a constant vector, is

(a) $\frac{\vec{F}}{2}$ (b) $-\frac{\vec{F}}{2}$ (c) $\vec{F} + \frac{30}{r^4}\vec{r}$ (d) $\vec{F} - \frac{30}{r^4}\vec{r}$

Q11. An infinite solenoid with its axis of symmetry along the z-direction carries a steady current *I*. The vector potential $\vec{A}$ at a distance *r* from the axis

(a) is constant inside and varies as *r* outside the solenoid

(b) varies as *r* inside and is constant outside the solenoid

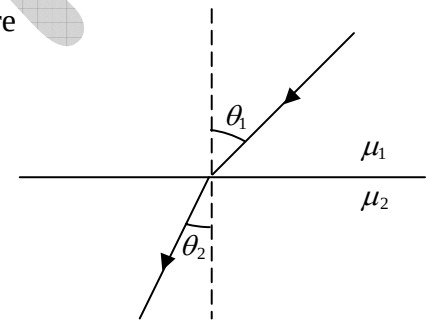
(c) varies as *r* inside and as $\frac{1}{r}$ outside the solenoid

(d) varies as $\frac{1}{r}$ inside and as *r* outside the solenoid

Q12. At the interface between two linear dielectrics (with permeability μ_1 and μ_2), the magnetic field lines bend, as shown in the figure. Assume that there is no free current at the interface. The ratio μ_1/μ_2 is:

(a) $\frac{\tan\theta_1}{\tan\theta_2}$ (b) $\frac{\cos\theta_1}{\cos\theta_2}$

(c) $\frac{\sin\theta_1}{\sin\theta_2}$ (d) $\frac{\cot\theta_1}{\cot\theta_2}$.



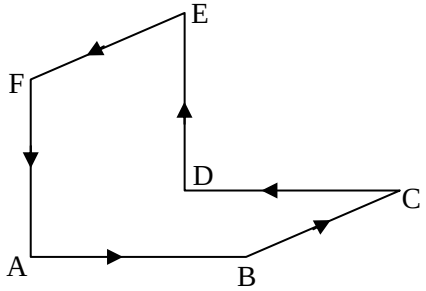
Q13. Two identical square frames are made of the same conducting wire such that ABCD plane is horizontal (with side AB in front) and ADEF plane is vertical, with side AD missing. The same current I flow in the frame as shown in figure. The direction of the magnetic moment vector will be

(a) vertical, upwards

(b) vertical, downwards

(c) at 45° with the vertical, pointing upwards

(d) at 45° with the vertical, pointing downwards

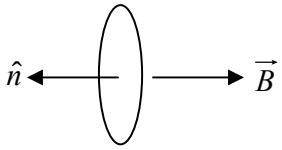


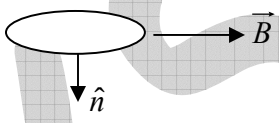
Q14. How does the magnetic moment (M) of an electron moving in a Bohr orbit of radius r_n in an atom depend upon r_n?

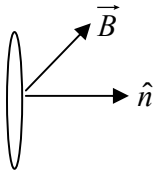
(a) M is directly proportional to r_n (b) M is directly proportional to $\sqrt{r_n}$

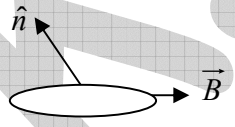
(c) M is directly proportional to $1/r_n$ (d) none of the above.

Q15. A current carrying loop is placed in a uniform magnetic field in four different orientations I, II, III and IV. Arrange them in the decreasing order of potential energy.

I. 

II. 

III. 

IV. 

(a) I > III > II > IV (b) I > IV > II > III

(c) I > II > III > IV (d) III > IV > I > II

Q16. A long circular cylinder of radius R carries a magnetization $\vec{M} = kr^2\hat{\phi}$ where k is a constant, r is the distance from the axis. Then the volume bound current and surface bound currents are respectively

(a) $2\pi kR^3, -2\pi kR^3$ (b) $-2\pi kR^3, 2\pi kR^3$

(c) $\pi kR^3, -\pi kR^3$ (d) $-\pi kR^3, \pi kR^3$

Q17. An infinitely long circular cylinder carries a uniform magnetization $\vec{M} = M\hat{z}$ parallel to its axis. Then the volume bound current $\left(\vec{J}_b\right)$ and surface bound currents $\left(\vec{K}_b\right)$ are respectively

(a) $M\hat{\phi}, 0$ (b) $0, M\hat{\phi}$

(c) $2M\hat{\phi}, 0$ (d) $0, 2M\hat{\phi}$

Q18. An infinitely long circular cylinder carries a uniform magnetization $\overrightarrow{M} = M\,\hat{z}$ parallel to its axis. Then which of the following statement is **not** true:

(a) the volume bound current $\left(\vec{J}_b\right)$ is zero

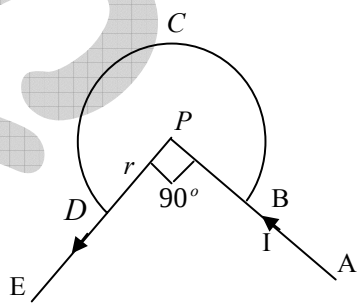
(b) the surface bound currents $\left(\vec{K}_b\right)$ is $M\hat{\phi}$

(c) the magnetic field $\left(\vec{B}\right)$ inside the cylinder is $\mu_0 \overrightarrow{M}$

(d) the magnetic field $\left(\vec{B}\right)$ inside the cylinder is zero

NAT (Numerical Answer Type)

Q19. A wire is given a shape $ABCDE$ as shown in the figure given below through which a steady current I flows. The magnetic field at the centre P is $\alpha\dfrac{\mu_0 I}{r}$. Then the value of α is……..(upto two decimal point)

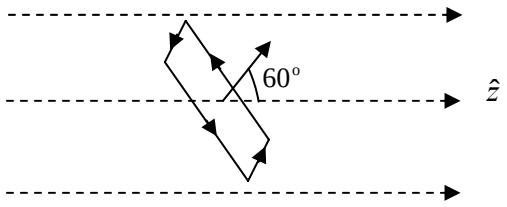


Q20. Current density in a cylindrical coordinate in certain region is given as

$$\vec{J} = \frac{5}{r}\hat{r} + \frac{10}{\left(r^2+1\right)}\hat{z}\ \ A/m^2.$$

The total current crossing the surface $z = 3,\ r \le 6$ in the $\hat{z}$ direction is…….. A

Q21. Consider a constant magnetic field of B Tesla along the z direction. A square loop of side length (L) meters is place in the field such that normal to the loop makes an angle of 60^0 to the z-axis, as shown in the figure. Then the value of the path integral $\oint \vec{A}\cdot\vec{dl}$ in the units of (Tesla m^2) is αBL^2. Then the value of α is …….



Q22. A particle of charge q and mass m moves in a circular orbit of radius r with angular speed ω. The ratio of the magnitude of its angular momentum to that of its magnetic moment is $\alpha\frac{m}{q}$. Then the value of α is …….

Q23. A small magnetic dipole is kept at the origin in the x-y plane. One wire L_1 is located at $z=-a$ in the x-z plane with a current I flowing in the positive x direction. Another wire L_2 is at $z=+a$ in y-z plane with the same current I as in L_1, flowing in the positive y-direction. The angle ϕ made by the magnetic dipole with respect to the positive x-axis is ……..$(in\ \text{deg}\,ree)$

MSQ (Multiple Select Questions)

Q24. Which of the following expressions for a vector potential $\vec{A}$ represent a uniform magnetic field of magnitude B_0 along the z-direction?

(a) $\vec{A}=(0,B_0x,0)$ (b) $\vec{A}=(-B_0y,0,0)$

(c) $\vec{A}=\left(\frac{B_0x}{2},\frac{B_0y}{2},0\right)$ (d) $\vec{A}=\left(-\frac{B_0y}{2},\frac{B_0x}{2},0\right)$

Q25. An infinitely long circular cylinder carries a uniform magnetization $\vec{M}=M\hat{z}$ parallel to its axis. Then which of the following statements are true:

(a) the volume bound current $\left(\vec{J}_b\right)$ is zero

(b) the surface bound currents $\left(\vec{K}_b\right)$ is $M\hat{\phi}$

(c) the magnetic field $\left(\vec{B}\right)$ inside the cylinder is $\mu_0\vec{M}$

(d) the magnetic field $\left(\vec{B}\right)$ inside the cylinder is zero

Q26. Two long parallel wires carrying current I_1 and I_2 in the same direction are placed a distance d apart. Then which of the following statements are true:

(a) the force per unit length on each wire is $\frac{\mu_0 I_1 I_2}{2\pi d}$

(b) the force per unit length on each wire is $\frac{\mu_0 I_1 I_2}{4\pi d}$

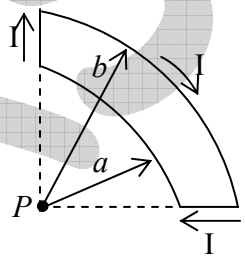
(c) they attract each other

(d) they repel each other

Q27. For the steady current configurations shown in figure which of the following statements are true:

(a) The magnetic field at point P is $B = \frac{\mu_0 I}{4}\left(\frac{1}{a} - \frac{1}{b}\right)$

(b) The magnetic field at point P is $B = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$



(c) Magnetic field points upward

(d) Magnetic field points inward

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (d)

Solution: The force on the two sides cancels.

At the bottom, $B=\frac{\mu_0 I_1}{2\pi a} \Rightarrow F=\left(\frac{\mu_0 I_1}{2\pi a}\right)I_2 a=\frac{\mu_0 I_1 I_2}{2\pi}$(up)

At the top, $B=\frac{\mu_0 I_1}{4\pi a} \Rightarrow F=\left(\frac{\mu_0 I_1}{4\pi a}\right)I_2 a=\frac{\mu_0 I_1 I_2}{4\pi}$(down)

Thus Net Force $=\frac{\mu_0 I_1 I_2}{4\pi}$(up)

Ans. 3: (c)

Ans. 4: (d)

Solution: The two half lines are the same as one infinite line: $\frac{\mu_0 I}{2\pi R}$

The half circle contributes: $\frac{\mu_0 I}{4R}$

So $B=\frac{\mu_0 I}{4R}\left(1+\frac{2}{\pi}\right)$ (into the page)

Ans. 5: (d)

Ans. 6: (c)

Solution: $\vec{B}=\vec{\nabla}\times\vec{A}=2(x-y)\hat{z} \Rightarrow \vec{J}=\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B})=-\frac{2}{\mu_0}(\hat{x}+\hat{y})$

Ans. 7: (c)

Solution: $\vec{B}\neq\nabla\times\vec{A}$.

Ans. 8: (c)

Solution: Surface current $\vec{K}=\sigma\vec{v} \Rightarrow \vec{K}=\sigma a\omega\hat{\phi}$, thus $\vec{A}=A_\phi\hat{\phi}$

Magnetic field inside is $\oint \vec{B}.d\vec{l}=\mu_0 I_{enc} \Rightarrow Bl=\mu_0 Kl \Rightarrow B=\mu_0 K \Rightarrow \vec{B}=\mu_0\sigma a\omega\hat{z}$

Since $\vec{B}=\vec{\nabla}\times\vec{A} \Rightarrow \oint_{line}\vec{A}.d\vec{l}=\int_S \vec{B}.d\vec{a} \Rightarrow |\vec{A}|\times 2\pi r=\mu_0\sigma a\omega\times\pi r^2 \Rightarrow |\vec{A}|=\frac{1}{2}\mu_0\sigma a\omega r$

Ans. 9: (b)

Solution: Surface current $\vec{K} = \sigma\vec{v} \Rightarrow \vec{K} = \sigma a\omega\hat{\phi}$, thus $\vec{A} = A_\phi\hat{\phi}$

Magnetic field inside is $\oint \vec{B}.d\vec{l} = \mu_0 I_{enc} \Rightarrow Bl = \mu_0 Kl \Rightarrow B = \mu_0 K \Rightarrow \vec{B} = \mu_0\sigma a\omega\hat{z}$

Since $\vec{B} = \vec{\nabla}\times\vec{A} \Rightarrow \oint\limits_{line} \vec{A}.d\vec{l} = \int\limits_S \vec{B}.d\vec{a} \Rightarrow \left|\vec{A}\right|\times 2\pi r = \mu_0\sigma a\omega\times\pi R^2 \Rightarrow \left|\vec{A}\right| = \frac{1}{2r}\mu_0\sigma a\omega R^2$

Ans. 10: (a)

Solution: $\vec{B} = \vec{\nabla}\times\vec{A} = \frac{1}{4}\left[\vec{\nabla}\times\left(\vec{F}\times\vec{r}\right)\right] + 10\left(\vec{\nabla}\times\frac{\vec{r}}{r^3}\right)$. Since $\vec{F}$ is a constant vector, let

$$\vec{F} = F_0\left(\hat{x}+\hat{y}+\hat{z}\right),\ \vec{F}\times\vec{r} = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ F_0 & F_0 & F_0\\ x & y & z\end{vmatrix} = \hat{x}(z-y)F_0 - \hat{y}(z-x)F_0 + \hat{z}(y-x)F_0$$

$$\vec{\nabla}\times\left(\vec{F}\times\vec{r}\right) = \begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ (z-y)F_0 & (x-z)F_0 & (y-x)F_0\end{vmatrix} = \hat{x}\left[F_0+F_0\right] - \hat{y}\left[-F_0-F_0\right] + \hat{z}\left[F_0+F_0\right] = 2F_0\left(\hat{x}+\hat{y}+\hat{z}\right)$$

$\Rightarrow \frac{1}{4}\left[\vec{\nabla}\times\left(\vec{F}\times\vec{r}\right)\right] = \frac{1}{2}F_0\left(\hat{x}+\hat{y}+\hat{z}\right) = \frac{\vec{F}}{2}$, $\vec{\nabla}\times\frac{\vec{r}}{r^3} = 0$. Thus $\vec{B} = \frac{\vec{F}}{2}$.

Ans. 11: (c)

Ans. 12: (a)

Solution: $\frac{\tan\theta_1}{\tan\theta_2} = \frac{B_1^{II}/B_1^{\perp}}{B_2^{II}/B_2^{\perp}} = \frac{B_2^{II}}{B_1^{II}} = \frac{\mu_1}{\mu_2}$ since $B_1^{\perp} = B_2^{\perp}, \mu_1 B_1^{II} = \mu_2 B_2^{II}$

Ans. 13: (c)

Ans. 14: (b)

Solution: $M = IA = \frac{ev_n}{2\pi r_n}\times\pi r_n^2 = \frac{ev_n r_n}{2}$ since $mv_n r_n = n\hbar \Rightarrow M \propto v_n r_n \propto n\hbar$

and $r_n = n^2 a_0 \Rightarrow n \propto \sqrt{r_n} \Rightarrow M \propto \sqrt{r_n}$

Ans. 15: (b)

Solution: $\because U = -\vec{m}.\vec{B} = -mB\cos\theta$

I. $\theta = 180^0 \Rightarrow U = +mB$, II. $\theta = 90^0 \Rightarrow U = 0$

III. $\theta = Acute\ angle \Rightarrow U = -ve$, IV. $\theta = Obtuse\ angle \Rightarrow U = +ve$

Thus I > IV > II > III

Ans. 16: (a)

Solution: $\vec{J}_b = \vec{\nabla}\times\vec{M} = \frac{1}{r}\frac{\partial}{\partial r}\left(r.kr^2\right)\hat{z} = 3kr\hat{z}$ and $\vec{K}_b = \vec{M}\times\hat{n} = kR^2\left(\hat{\phi}\times\hat{r}\right) = -kR^2\hat{z}$

So the bound current flows up the cylinder, and returns down the surface.

Total volume bound current $= \int \vec{J}_b.d\vec{a} = \int J_b da = \int_0^R (3kr)2\pi r dr = 2\pi kR^3$

Total surface bound current $= \int K_b dl = -kR^2 \times 2\pi R = -2\pi kR^3$

Ans. 17: (b)

Solution: $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$, $\vec{K}_b = \vec{M}\times\hat{n} = M\left(\hat{z}\times\hat{r}\right) = M\hat{\phi}$

Ans. 18: (d)

Solution: $\vec{J}_b = \vec{\nabla}\times\vec{M} = 0$, $\vec{K}_b = \vec{M}\times\hat{n} = M\left(\hat{z}\times\hat{r}\right) = M\hat{\phi}$

The field is that of a surface current $\vec{K}_b = M\hat{\phi}$, that is the case of a **solenoid**,

So the field outside is zero.

Field inside is: $B = \mu_0 K_b = \mu_0 M \Rightarrow \vec{B} = \mu_0\vec{M}$

NAT (Numerical Answer Type)

Ans. 19: **0.38**

$$B=\frac{\mu_0 I}{2r}\times\frac{\theta}{2\pi}=\frac{\mu_0 I}{2r}\times\frac{270^0}{360^\circ}=\frac{3\mu_0 I}{8r}=0.38\frac{\mu_0 I}{r}$$

Ans. 20: **113.38**

Solution: $I=\int\vec{J}.d\vec{a}=\iint\left[\frac{5}{r}\hat{r}+\frac{10}{r^2+1}\hat{z}\right].rdrd\phi\hat{z}$

$$I=\int_0^{2\pi}\int_0^b\frac{10}{r^2+1}rdrd\phi=2\pi\times\frac{1}{2}\times10\ln\left(b^2+1\right)\Rightarrow I=10\pi\ln\left(37\right)=31.4\times3.6=113.38\ A$$

Ans. 21: **0.5**

Solution: $\oint\vec{A}\cdot\vec{dl}=\int_S\left(\vec{\nabla}\times\vec{A}\right).d\vec{a}=\int_S\vec{B}.d\vec{a}=BA\cos60^0=B\times L^2\times\frac{1}{2}=\frac{1}{2}BL^2=0.5BL^2$

Ans. 22: **2**

Solution: $M=IA=\frac{q}{T}A=\frac{q\omega}{2\pi}\times\pi r^2=\frac{q\omega r^2}{2}$ and $L=mvr=mr^2\omega\Rightarrow\frac{L}{M}=2\frac{m}{q}$

Ans. 23: **225**

Solution: Magnetic field at $z=0$ due to wire at $z=-a$ is $\vec{B}=-B\hat{y}$.

Magnetic field at $z=0$ due to wire at $z=+a$ is $\vec{B}=-B\hat{x}$.

Resultant magnetic field at $z=0$ makes an angle of 45^0 with $-\hat{x}$ and 225^0 with $\hat{x}$.

MSQ (Multiple Select Questions)

Ans. 24: (a), (b) and (d)

Ans. 25: (a), (b) and (c)

Solution: $\vec{J}_b = \vec{\nabla} \times \vec{M} = 0$, $\vec{K}_b = \vec{M} \times \hat{n} = M\left(\hat{z} \times \hat{r}\right) = M\hat{\phi}$

The field is that of a surface current $\vec{K}_b = M\hat{\phi}$, that is the case of a **solenoid**,

So the field outside is zero.

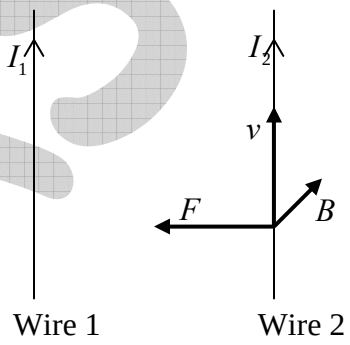
Field inside is: $B = \mu_0 K_b = \mu_0 M \Rightarrow \vec{B} = \mu_0 \vec{M}$

Ans. 26: (a) and (c)

Solution: The field at (2) due to (1) is $B = \frac{\mu_0 I_1}{2\pi d}$ (points into the page).

Force on (2) is $F = I_2 \times \frac{\mu_0 I_1}{2\pi d} \int dl \Rightarrow f = \frac{\mu_0 I_1 I_2}{2\pi d}$

Force per unit length is towards (1) and net force is attractive



Ans. 27: (b) and (c)

Solution: The straight segment produces no field at P.

The two quarter circles give $B = \frac{\mu_0 I}{8}\left(\frac{1}{a} - \frac{1}{b}\right)$ out of page.

Chapter - 3

Dynamics of Charged Particles in Static and Uniform Electromagnetic Fields

The Lorentz Force Law:

The magnetic force on a charge Q, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is

$$\vec{F}_{mag} = Q\left(\vec{v}\times\vec{B}\right)$$

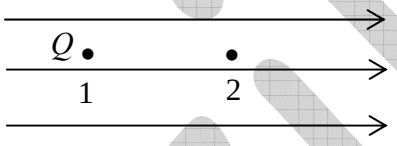
This is known as Lorentz force law.

In the presence of both electric and magnetic fields, the net force on Q would be:

$$\vec{F} = Q\left[\vec{E}+\left(\vec{v}\times\vec{B}\right)\right]$$

3.1 Charged Particle in Static Electric Field

3.1.1 Charged Particle enters in the direction of field (Linear motion)



The force on the charge Q in electric field $\vec{E}$ is $\vec{F} = Q\vec{E}$.

Acceleration of the charge particle in the direction of the electric field is $\vec{a} = \frac{\vec{F}}{m} = \frac{Q\vec{E}}{m}$.

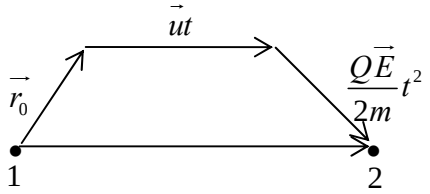
If $\vec{r}$ is the position vector at any time t then

$\frac{d^2\vec{r}}{d^2t} = \frac{Q}{m}\vec{E} \Rightarrow \frac{d\vec{r}}{dt} = \frac{Q\vec{E}}{m}t + C \Rightarrow \frac{d\vec{r}}{dt} = \vec{v} = \frac{Q\vec{E}}{m}t + C$ where C is a constant.

Let at $t = 0$, $\vec{v} = \vec{u} \Rightarrow C = \vec{u} \Rightarrow \vec{v} = \frac{Q\vec{E}}{m}t + \vec{u}$

Since $\frac{d\vec{r}}{dt} = \frac{Q\vec{E}}{m}t + \vec{u} \Rightarrow \vec{r} = \frac{Q\vec{E}}{2m}t^2 + \vec{u}t + C_1$ where C_1 is a constant

Let at $t = 0$, $\vec{r} = \vec{r_0} \Rightarrow C_1 = \vec{r_0} \Rightarrow \vec{r} = \frac{Q\vec{E}}{2m}t^2 + \vec{u}t + \vec{r_0}$



If initially $\vec{u} = 0, \vec{r_0} = 0 \Rightarrow \vec{r} = \frac{Q\vec{E}}{2m}t^2$ and $\vec{v} = \frac{Q\vec{E}}{m}t$.

The energy acquired by the charged particle in moving from point 1 to 2 is

$$W=\int_1^2 \vec{F}.d\vec{l}=m\int_1^2 \vec{a}.d\vec{l}=m\int_1^2 \frac{d\vec{v}}{dt}.\vec{v}dt=m\int_1^2 \vec{v}.d\vec{v}\Rightarrow W=\frac{1}{2}m\left(v_2^2-v_1^2\right)$$

If the potential difference between points 1 to 2 is V then

$$W=QV=\frac{1}{2}m\left(v_2^2-v_1^2\right)$$

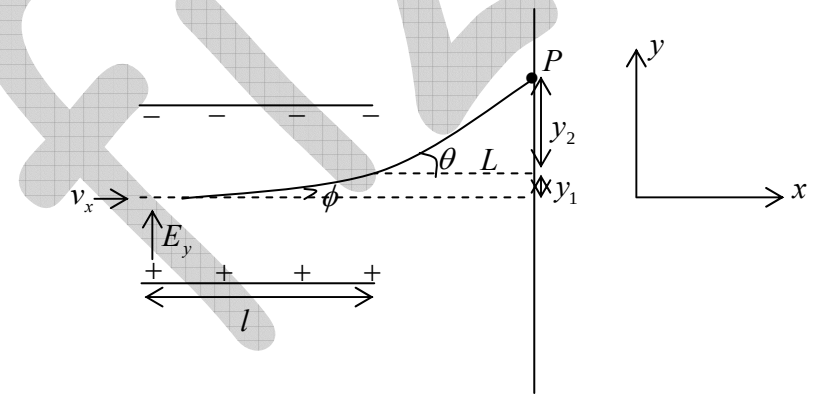
If the particle starts from rest i.e $\left(v_1=0\right)$ and final velocity is v then

$$W=QV=\frac{1}{2}mv^2\Rightarrow v=\sqrt{\frac{2QV}{m}}$$

Kinetic energy of the particle $K.E.=\frac{1}{2}mv^2=\frac{1}{2}m.\frac{Q^2E^2}{m^2}t^2=QE\times\frac{QE}{2m}t^2=QEr$

3.1.2 Charged Particle enters in the direction perpendicular to field (Parabolic motion)

Let us consider a charge particle enters in an electric field region with velocity v_x at $t=0$. The electric field is in the y-direction and the field region has length l. After traversing a distance l it strikes a point P on a screen which is placed at a distance L from the field region.



Since electric field is in the y-direction, charge particle will experience force

$$F_y=QE_y\Rightarrow \text{Acceleration } a_y=\frac{QE_y}{m}$$

In time *t*, charge particle will traverse a distance $y = \frac{1}{2} a_y t^2$ in *y*-direction and a distance $x = v_x t$ in *x*-direction.

Thus $y = \frac{1}{2} a_y t^2 = \frac{QE_y}{2m}\left(\frac{x}{v_x}\right)^2$ and which represents ***parabolic path***.

$y_1 = \frac{QE_y}{2m}\left(\frac{l}{v_x}\right)^2$ and $y_2 == L\tan\theta$

Thus distance of point P from the center of the screen is, $y_1 + y_2 = \frac{QE_y}{2m}\left(\frac{l}{v_x}\right)^2 + L\tan\theta$

Angle of deviation in the field region, $\tan\phi = \frac{dy}{dx} = \frac{QE_y}{mv_x^2}x$

Angle of deviation in the field free region, $\tan\theta = \frac{QE_y}{mv_x^2}l$

3.2 Charged Particle in Static Magnetic Field

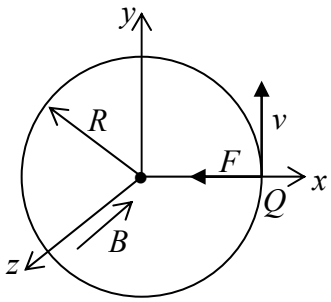
The magnetic force on a charge *Q*, moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is,

$$\vec{F}_{mag} = Q\left(\vec{v}\times\vec{B}\right)$$

This is known as Lorentz force law.

3.2.1 Charged Particle enters in the direction perpendicular to field (Circular motion)

If a charge particle enters in a magnetic field at angle of 90°, then motion will be circular with the magnetic force providing the centripetal acceleration. As shown in figure, a uniform magnetic field points into the page; if the charge *Q* moves counter clockwise, with speed *v*, around a circle of radius *R*, the magnetic force points inward, and has a fixed magnitude *QvB*, just right to sustain uniform circular motion:



$$QvB = m\frac{v^2}{R} \Rightarrow R = \frac{mv}{QB}$$

where R is the radius of the circle and m is the mass of the charge particle.

Momentum of the charged particle $p = QBR$

Kinetic energy (K.E) $= \frac{p^2}{2m} = \frac{Q^2B^2R^2}{2m}$

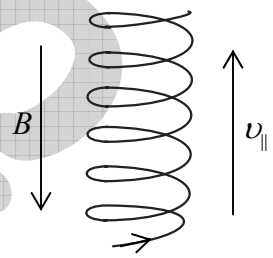
Time period $T = \frac{2\pi R}{v} = \frac{2\pi m}{QB}$

3.2.2 Charged Particle enters in the direction making an angle with the field (Helical motion)

If the charge particle enters in a magnetic field making an angle θ, then motion will be helical.

$v_\perp = v\sin\theta$ and $v_\parallel = v\cos\theta$,

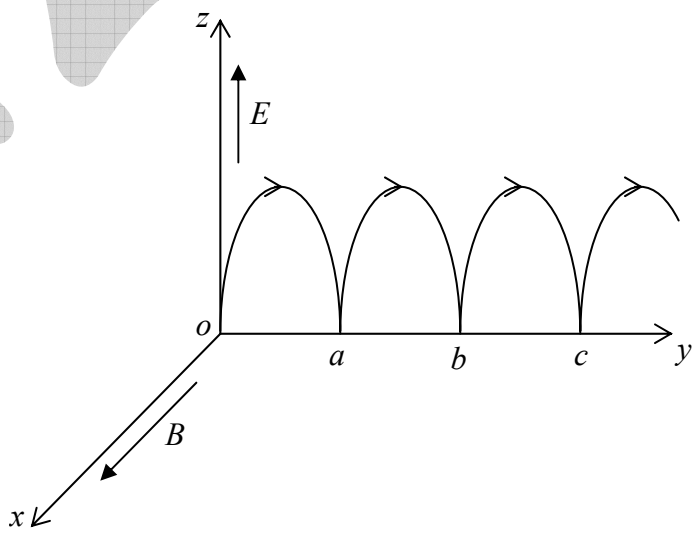
and the radius of helix is $R = \frac{mv_\perp}{QB}$.



3.3 Charged Particle in Uniform Electric and Magnetic Field (Cycloid motion)

If $\vec{B}$ points in x-direction and $\vec{E}$ points in z-direction, and a particle at rest is released from origin, then particle will follow cycloid motion.

Initially, the particle is at rest, so the magnetic force is zero, and the electric field accelerates the charge in z-direction. As it speeds up, a magnetic force develops which pulls the charge to the right. The faster it goes stronger the magnetic force becomes and it curves the particle back around towards the



y-axis. At this point the charge is moving against the electric force, so it begins to slow down-the magnetic force then decreases, and the electrical force takes over, bringing the charge to rest at point a and then process repeats.

$$\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = Q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right) = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$$

$$\Rightarrow \ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right) \quad where\ \omega = \frac{QB}{m}(cylotron\ frequency)$$

Let us solve the above differential equations,

$$\because \ddot{y} = \omega\dot{z} \Rightarrow \dddot{y} = \omega\ddot{z} \Rightarrow \dddot{y} = \omega^2\left(\frac{E}{B} - \dot{y}\right) \qquad \because \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$$

$$\Rightarrow \dddot{y} + \omega^2\dot{y} = \omega^2\frac{E}{B}$$

Let $\dot{y} = t' \Rightarrow \ddot{t}' + \omega^2 t' = \omega^2\frac{E}{B}$.

For $C.F.$

$$D^2 + \omega^2 = 0 \Rightarrow D = \pm i\omega \Rightarrow C.F. = C_1'\cos\omega t + C_2'\sin\omega t$$

For $P.I.$

$$P.I. = \frac{1}{D^2 + \omega^2}\left(\omega^2\frac{E}{B}\right) = \frac{1}{0^2 + \omega^2}\left(\omega^2\frac{E}{B}\right) = \frac{E}{B}$$

Thus $t' = \dot{y} = C.F. + P.I = C_1'\cos\omega t + C_2'\sin\omega t + \frac{E}{B}$

$$\Rightarrow y(t) = C_1\sin\omega t + C_2\cos\omega t + \frac{E}{B}t + C_3 \quad \text{where } C_1 = \frac{C_1'}{\omega}, C_2 = -\frac{C_2'}{\omega}.$$

$$\because \omega\dot{z} = \ddot{y} \Rightarrow \dot{z} = -C_1'\sin\omega t + C_2'\cos\omega t \Rightarrow z(t) = C_1\cos\omega t - C_2\sin\omega t + C_4$$

Initial Condition: $\dot{y}(0) = 0, \dot{z}(0) = 0$ *and* $y(0) = 0, z(0) = 0$

$$\because y(0) = 0, z(0) = 0 \Rightarrow y(0) = C_2 + C_3 = 0 \text{ and } \Rightarrow z(0) = C_1 + C_4 = 0$$

$$\because \dot{y}(0) = 0, \dot{z}(0) = 0 \Rightarrow \dot{y}(0) = \omega C_1 + \frac{E}{B} = 0 \text{ and } \Rightarrow \dot{z}(0) = -C_2\omega = 0 \Rightarrow C_2 = 0$$

Thus $C_2 = 0 \Rightarrow C_3 = 0$ and $\Rightarrow C_1 = -C_4 = -\dfrac{E}{\omega B}$

$$y(t) = \frac{E}{\omega B}(\omega t - \sin \omega t),\ z(t) = \frac{E}{\omega B}(1 - \cos \omega t)$$

$$\Rightarrow (y - R\omega t)^2 + (z - R)^2 = R^2 \qquad where\ R = \frac{E}{\omega B}$$

This is the formula for a circle**,** of radius *R*, whose center is $(0, R\omega t, R)$ travels in the *y*-direction at constant speed, $v = \omega R = \dfrac{E}{B}$

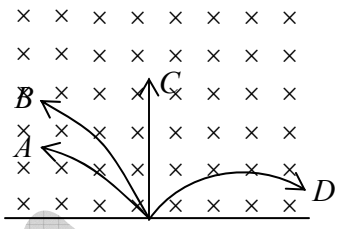
The curve generated in this way is called a cycloid.

Magnetic forces do not work because $(\vec{v} \times \vec{B})$ is perpendicular to $\vec{v}$, so

$$dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q(\vec{v} \times \vec{B}) \cdot \vec{v} dt = 0.$$

Magnetic forces may alter the direction in which a particle moves, but they can not speed up or slow down it.

Example: A neutron, a proton, an electron and an α – particle enter a region of constant magnetic field with equal velocities. The magnetic field is along the inward normal to the plane of paper. Label the tracks of the particles.

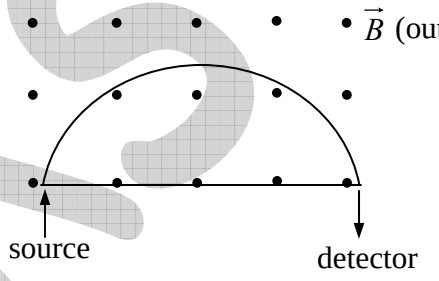


Solution: $R_e = -\frac{m_e v}{eB}$, $R_p = \frac{m_p v}{eB} = 1836\, R_e$

$$R_\alpha = \frac{m_\alpha v}{q_\alpha B} = \frac{4}{2}\frac{m_p v}{eB} = 2R_p .$$

A → Proton, B → α–particle, C → neutron (undeflected), D → electron.

Example: A uniform magnetic field with a slit system, as shown in figure, is to be used as a momentum filter for high energy charged particles. With a field B tesla, it is found that the filter transmits α–particles each of energy 5.3 MeV.



The magnetic field is increased to 2.3 B tesla and deuterons are passed into the filter. Find the energy of each deuteron transmitted by the filter.

Solution: $r = \frac{mv}{qB} \Rightarrow mv = qBr \Rightarrow p = qBr \Rightarrow p^2 = 2mK = (qBr)^2$

$$\Rightarrow \frac{2m_\alpha K_\alpha}{2m_d K_d} = \left(\frac{q_\alpha \times r_\alpha \times B}{q_d \times r_d \times 2\cdot 3B}\right)^2 \Rightarrow \frac{2}{2}.\frac{4m_p}{2m_p} \times \frac{5\cdot 3}{K_d} = \left(\frac{2e}{e} \times \frac{r}{r} \times \frac{1}{2\cdot 3}\right)^2 \Rightarrow K_d = 14\ MeV$$

Example: A beam of protons with velocity $4\times10^5\,\mathrm{m/sec}$ enters a uniform magnetic field of 0.3 Tesla at an angle of 60^0 to the magnetic field. Find the radius of the helical path taken by the proton beam. Also find the pitch of the helix.

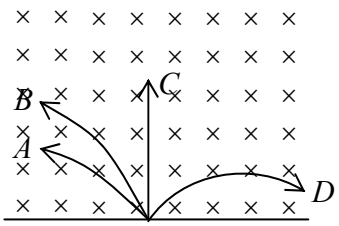
Solution: $v = 4\times10^5 m/\sec$, $v_\perp = v\sin 60$ and $v_\parallel = v\cos 60$

$$\frac{m_p v_\perp^2}{R} = q v_\perp B \Rightarrow R = \frac{m_p v_\perp}{qB} = 0.012\, m \text{ where } \mathrm{m_p} = 1836\mathrm{m_e}.$$

Pitch of helix $d = v_\parallel T = v_\parallel \times \frac{2\pi m_p}{qB} = 0.044\, m$

MCQ (Multiple Choice Questions)

Q1. A neutron, a proton, an electron and an α – particle enter a region of constant magnetic field with equal velocities. The magnetic field is along the inward normal to the plane of paper. Tracks of the particles are labeled as shown in figure. Which of the following correctly represents the track of the particles.



(a) A → Proton, B → α–particle, C → neutron, D → electron

(b) A → α–particle, B → Proton, C → neutron, D → electron

(c) A → electron, B → α–particle, C → neutron, D → Proton

(d) A → Proton, B → electron, C → neutron, D → α–particle

Q2. In a cyclotron, α-particles are accelerated using RF source of frequency $15\,MHz$. What would be the frequency of RF source if α-particles are replaced by ${}_2He^3$ particles?

(a) $9\,MHz$ (b) $12\,MHz$ (c) $16\,MHz$ (d) $20\,MHz$

Q3. A proton, deuteron and an α-particle having the same kinetic energy are moving in circular trajectories in a constant magnetic field. If r_p, r_d and r_α denote respectively the radii of the trajectories of these particles, then

(a) $r_\alpha = r_p < r_d$ (b) $r_\alpha = r_p > r_d$ (c) $r_\alpha > r_p < r_d$ (d) $r_\alpha = r_p = r_d$

Q4. If $\vec{B}$ points in x-direction and $\vec{E}$ points in z-direction, and a particle of mass m and charge Q at rest is released from origin, then particle will follow Cycloidal path. Which of the following coupled differential equation represents Cycloidal path ($where\ \omega = \frac{QB}{m}$)

(a) $\ddot{x} = \omega\dot{y},\ \ddot{y} = \omega\left(\frac{E}{B} - \dot{x}\right)$ (b) $\ddot{x} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{x}\right)$

(c) $\ddot{z} = \omega\dot{y},\ \ddot{y} = \omega\left(\frac{E}{B} - \dot{z}\right)$ (d) $\ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$

Q5. A proton, deuteron and an α-particle having the same kinetic energy are moving in circular trajectories in a constant magnetic field. If r_p, r_d and r_α denote respectively the radii of the trajectories of these particles, then the ratio $r_p : r_d : r_\alpha$ is

(a) $1:1:\sqrt{2}$ (b) $1:\sqrt{2}:1$ (c) $\sqrt{2}:1:1$ (d) $1:1:1$

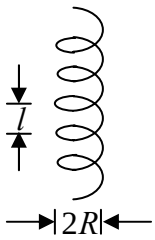
Q6. A particle having a charge Q and mass m moves along a circle of radius R under the action of a magnetic field B. When the particle is at a point P, a uniform electric field is switched on and it is found that the particle continues on the tangent through P with a uniform velocity. The magnitude of the electric field is:

(a) $\frac{m}{QB}$ (b) $\frac{RQB}{m}$ (c) $\frac{RQB^2}{m}$ (d) RQB^2.

Q7. Two particles X and Y having equal charges, after being accelerated through the same potential difference, enter a region of uniform magnetic field and describe circular paths of radii R_1 and R_2 respectively. The ratio of the mass of X to that Y is

(a) $\sqrt{\frac{R_1}{R_2}}$ (b) $\sqrt{\frac{R_2}{R_1}}$ (c) $\left(\frac{R_1}{R_2}\right)^2$ (d) $\frac{R_1}{R_2}$

Q8. A charged particle moves in a helical path under the influence of a constant magnetic field. The initial velocity is such that the component along the magnetic field is twice the component in the plane normal to the magnetic field. The ratio ℓ / R of the pitch ℓ to the radius R of the helical path is



(a) $\pi/2$ (b) 4π (c) 2π (d) π

NAT (Numerical Answer Type)

Q9. The maximum energy of deuteron coming out of a cyclotron accelerator is $20\,MeV$. The maximum energy of protons that can be obtained from the accelerator is …….. MeV

Q10. In a cyclotron, α-particles are accelerated using RF source of frequency $12\,MeV$. The frequency of RF source if α-particles are replaced by ${}_2He^3$ particles…….. MeV

MSQ (Multiple Select Questions)

Q11. If $\vec{B}$ points in x-direction and $\vec{E}$ points in z-direction, and a particle of mass m and charge Q at rest is released from origin, then particle will follow Cycloidal path. Which of the following expressions are true for Cycloidal path (*where* $\omega = \frac{QB}{m}$ and $R = \frac{E}{B}$)

(a) $\ddot{z} = \omega\dot{y},\ \ddot{y} = \omega\left(\frac{E}{B} - \dot{z}\right)$

(b) $\ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$

(c) $\left(y - R\omega t\right)^2 + z^2 = R^2$

(d) $\left(z - R\omega t\right)^2 + y^2 = R^2$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (a)

Solution: $R_e = -\frac{m_e v}{eB}, R_p = \frac{m_p v}{eB} = 1836 R_e,\ R_\alpha = \frac{m_\alpha v}{q_\alpha B} = \frac{4m_p v}{2eB} = 2R_p$

Ans. 2: (d)

Solution: $f = \frac{qB}{2\pi m} \Rightarrow \frac{f'}{f_\alpha} = \frac{q'}{q_\alpha} \times \frac{m_\alpha}{m'} = \frac{q}{q} \times \frac{4m_p}{3m_p} = \frac{4}{3} \Rightarrow f' = 20\ MHz$

Ans. 3: (a)

Solution: $r = \frac{mv}{qB} \Rightarrow r = \frac{\sqrt{2mK}}{qB} \Rightarrow r_p : r_d : r_\alpha = \frac{\sqrt{1}}{1} : \frac{\sqrt{2}}{1} : \frac{\sqrt{4}}{2}$

$\Rightarrow r_p : r_d : r_\alpha = 1 : \sqrt{2} : 1$ or $r_\alpha = r_p < r_d$

Ans. 4: (d)

Solution: $\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = Q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right) = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$

$\Rightarrow \ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$ where $\omega = \frac{QB}{m}$ (cylotron frequency)

Ans. 5: (b)

Solution: $r = \frac{mv}{qB} \Rightarrow r = \frac{\sqrt{2mK}}{qB} \Rightarrow r_p : r_d : r_\alpha = \frac{\sqrt{1}}{1} : \frac{\sqrt{2}}{1} : \frac{\sqrt{4}}{2}$

$\Rightarrow r_p : r_d : r_\alpha = 1 : \sqrt{2} : 1$.

Ans. 6: (c)

Solution: $QE = QvB \Rightarrow E = vB$ and $R = \frac{mv}{QB} \Rightarrow E = \frac{RQB^2}{m}$.

Ans. 7: (c)

Solution: $K_X = \frac{1}{2} m_1 v_1^2 = qV$ and $K_Y = \frac{1}{2} m_2 v_2^2 = qV$ $\Rightarrow m_1 v_1^2 = m_2 v_2^2$ …(1)

The magnetic force provides centripetal force to the charged particle

$$\therefore \frac{m_1 v_1^2}{R_1} = q v_1 B;\ \ \frac{m_2 v_2^2}{R_2} = q v_2 B \ \Rightarrow \frac{m_1 v_1}{R_1} = \frac{m_2 v_2}{R_2} \qquad \ldots(2)$$

From (1) and (2), $\frac{m_1}{m_2} = \left(\frac{R_1}{R_2}\right)^2$

Ans. 8: (b)

Solution: $v_{\parallel} = 2v_{\perp}$

Pitch of the helix $l = v_{\parallel} T = v_{\parallel} \frac{2\pi R}{v_{\perp}} = 2v_{\perp} \frac{2\pi R}{v_{\perp}} = 4\pi R \Rightarrow \frac{l}{R} = 4\pi$

NAT (Numerical Answer Type)

Ans. 9: 40

Solution: $KE_{\max} = \frac{q^2 B^2 r^2}{2m} \Rightarrow \frac{k_p}{k_d} = \left(\frac{q_p}{q_d}\right)^2 \times \frac{m_d}{m_p} = \left(\frac{q}{q}\right)^2 \times \frac{2m_p}{m_p} \Rightarrow k_p = 2k_d = 40\ MeV$

Ans. 10: 16

Solution: $f = \frac{qB}{2\pi m} \Rightarrow \frac{f'}{f_\alpha} = \frac{q'}{q_\alpha} \times \frac{m_\alpha}{m'} = \frac{q}{q} \times \frac{4m_p}{3m_p} = \frac{4}{3} \Rightarrow f' = 16\ MHz$

MSQ (Multiple Select Questions)

Ans. 11: (b) and (c)

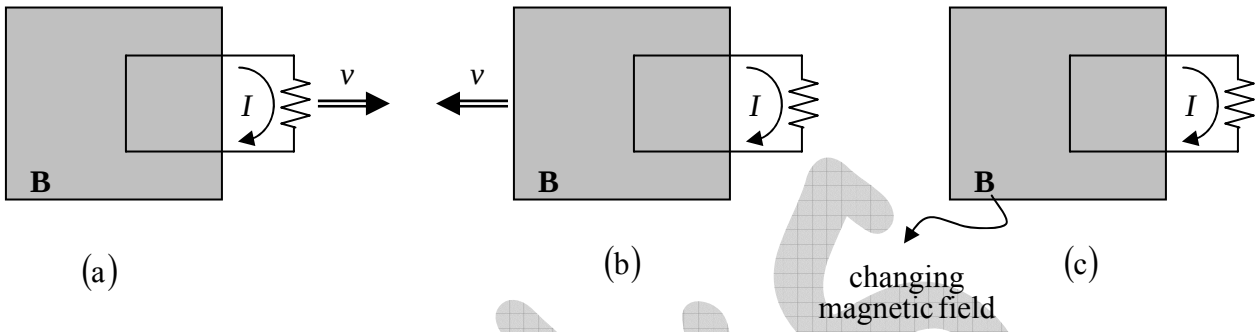
Solution: $\vec{F} = Q\left(\vec{E} + \vec{v} \times \vec{B}\right) = Q\left(E\hat{z} + B\dot{z}\hat{y} - B\dot{y}\hat{z}\right) = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right)$

$\Rightarrow \ddot{y} = \omega\dot{z},\ \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right)$ where $\omega = \frac{QB}{m}$ (cylotron frequency)

Chapter - 4

Electromagnetic Induction

4.1 Faraday's Law



Experiment 1: He pulled a loop of wire to the right through a magnetic field. A current flowed in the loop (Figure a).

Experiment 2: He moved the magnet to the left, holding the loop still. Again, a current flowed in the loop (Figure b).

Experiment 3: With both the loop and the magnet at rest, he changed the strength of the field (he used an electromagnet, and varied the current in the coil). Once again current flowed in the loop (Figure c).

Thus, universal flux rule is that, whenever (and for whatever reason) the magnetic flux through a loop changes, an e.m.f. (ε) will appear in the loop $\varepsilon = -\frac{d\Phi}{dt}$

In experiment 2, *A changing magnetic field induces an electric field.*

It is this "induced" electric field that accounts for the e.m.f.

Also the induced e.m.f $\varepsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$ (where magnetic flux $\Phi = \int \vec{B}.d\vec{a}$)

Then $\vec{E}$ is related to the change in $\vec{B}$ by the equation

$$\oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}.d\vec{a} \qquad \Rightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

4.1.1 Lenz's Law

In Faraday's law negative sign represents the **Lenz's law.** (The induced current will flow in such a direction that the flux it produces tends to cancel the change).

For example if the magnetic flux is increasing then induced e.m.f will try to reduce and vice versa.

Example: A long solenoid, of radius *a*, is driven by alternating current, so that the field inside is sinusoidal $\vec{B}(t) = B_0 \cos(\omega t)\hat{z}$. A circular loop of wire, of radius $a/2$ and resistance *R*, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.

Solution: Magnetic flux through the loop $\phi = \vec{B}.\vec{A} = B_0 \cos(\omega t) \times \frac{\pi a^2}{4} = \frac{1}{4}\pi a^2 B_0 \cos(\omega t)$

Induced emf $\varepsilon(t) = -\frac{d\phi}{dt} = \frac{1}{4}\pi a^2 B_0 \omega \sin \omega t = \varepsilon_0 \sin \omega t \Rightarrow \varepsilon_0 = \frac{1}{4}\pi a^2 B_0 \omega$

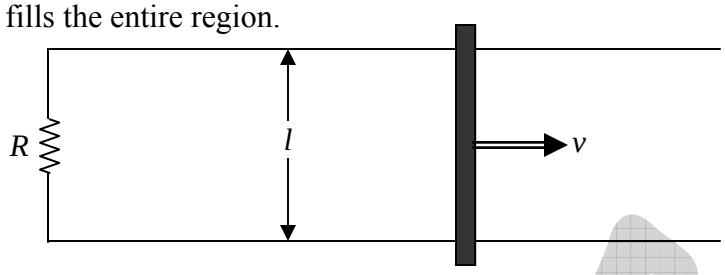
Induced current $i(t) = \frac{\varepsilon(t)}{R} = \frac{\pi a^2 B_0 \omega \sin \omega t}{4R} = i_0 \sin \omega t \Rightarrow i_0 = \frac{\pi a^2 B_0 \omega}{4R}$

Example: A square loop (side *a*) is mounted on a vertical shaft and rotated at angular velocity ω. A uniform magnetic field $\vec{B}$ points to the right. Find the induced emf $\varepsilon(t)$ for this alternating current generator.

Solution: Magnetic flux $\phi = \vec{B}.\vec{A} = BA\cos\theta = Ba^2 \cos \omega t$

Induced emf $\varepsilon(t) = -\frac{d\phi}{dt} = Ba^2 \omega \sin \omega t$

Example: A metal bar of mass *m* slides frictionlessly on two parallel conducting rails a distance *l* apart. A resistor *R* is connected across the rails and a uniform magnetic field $\vec{B}$, pointing into the page, fills the entire region.



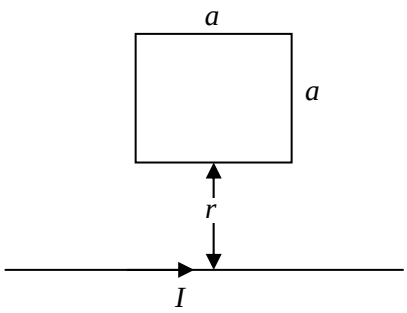
(a) If the bar moves to the right at speed *v*, what is the current in the resistor? In what direction does it flow?

(b) What is the magnetic force on the bar?

Solution: (a) $\varepsilon = -\frac{d\Phi}{dt} = -Bl\frac{dx}{dt} = -Blv$; $\varepsilon = IR \Rightarrow I = \frac{Blv}{R}$ (downward in R).

(b) $F = IlB = \frac{B^2 l^2 v}{R}$ (to the left)

Example: A square loop of wire (side *a*) lies on a table, a distance *r* from a very long straight wire, which carries a current *I*.



(a) Find the flux of $\vec{B}$ through the loop.

(b) If some one now pulls the loop directly away from the wire, at speed *v*, what emf is generated? In what direction does the current flow?

(c) What if the loop is pulled to the right at speed *v*, instead of moving away?

Solution: (a) $\Phi = \int \vec{B}.d\vec{a} = \frac{\mu_0 I}{2\pi}\int_r^{r+a}\frac{1}{r}(adr) = \frac{\mu_0 I}{2\pi}\ln\left(\frac{r+a}{r}\right)$

(b) $\varepsilon = -\frac{d\Phi}{dt} = -\frac{\mu_0 Ia}{2\pi}\frac{d}{dt}\ln\left(\frac{r+a}{r}\right)$

$\Rightarrow \varepsilon = -\frac{\mu_0 Ia}{2\pi}\left(\frac{1}{r+a}\frac{dr}{dt} - \frac{1}{r}\frac{dr}{dt}\right) = \frac{\mu_0 Ia^2 v}{2\pi r(r+a)}$ (*counter clockwise*)

(c) Flux is constant so $\varepsilon = 0$.

Example: A long solenoid, of radius *a* and *n* turns per unit length carries a time-dependent current $I(t)$ in the $\hat{\phi}$ direction. Find the electric field (magnitude and direction) at a distance *r* from the axis (both inside and outside the solenoid).

Solution: Field due to solenoid $\vec{B}(t) = \mu_0 n I(t)\hat{z}$ *inside* and zero outside.

Inside solenoid ($r < a$):

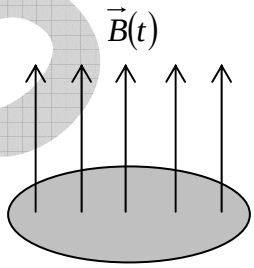
$$\oint \vec{E}.d\vec{l} = -\frac{d}{dt}\int \vec{B}.d\vec{a} \Rightarrow \left|\vec{E}\right|(2\pi r) = -\mu_0 n \frac{dI(t)}{dt} \times \pi r^2 \Rightarrow \vec{E} = -\mu_0 n \frac{r}{2}\frac{dI(t)}{dt}\hat{\phi}.$$

Outside solenoid ($r > a$):

$$\oint \vec{E}.d\vec{l} = -\frac{d}{dt}\int \vec{B}.d\vec{a} \Rightarrow \left|\vec{E}\right|(2\pi r) = -\mu_0 n \frac{dI(t)}{dt} \times \pi a^2$$

$$\Rightarrow \vec{E} = -\frac{\mu_0 n a^2}{2r}\frac{dI(t)}{dt}\hat{\phi}.$$

Example: A uniform magnetic field $\vec{B}(t)$, pointing straight up, fills the shaded circular region of figure shown below. If $\vec{B}$ is changing with time, what is the induced electric field?



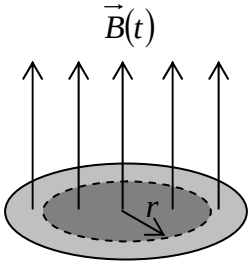
Solution: $\vec{E}(t)$ points in the circumferential direction, just like the magnetic field inside a long straight wire carrying a uniform current density.

Draw an Amperian loop of radius *r* and apply Faraday's Law:

$$\oint \vec{E}.d\vec{l} = -\frac{d\phi}{dt} \Rightarrow \left|\vec{E}\right|(2\pi r) = -\frac{d}{dt}\left(\pi r^2 B(t)\right) = -\pi r^2 \frac{dB(t)}{dt}.$$

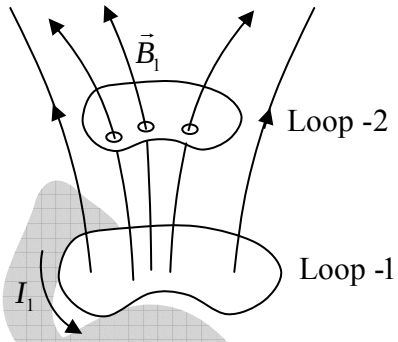
Thus $$\vec{E} = -\frac{r}{2}\frac{dB(t)}{dt}\hat{\phi}.$$

If $\vec{B}$ is increasing, $\vec{E}$ runs clockwise, as viewed from above.



4.1.2 Inductance

If a steady current I_1 flows in a loop 1, it produces magnetic field $\vec{B_1}$. Some of the field lines pass through loop 2, let Φ_2 be the flux of $\vec{B_1}$ through 2.



From Biot-Savart law, $\vec{B_1} = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l}_1 \times \hat{R}}{R^2}$,

Therefore flux through loop 2 is $\Phi_2 = \int \vec{B_1} \cdot d\vec{a_2}$.

Thus $\Phi_2 = M_{21} I_1$, where M_{21} is the constant of proportionality; it is known as the **mutual inductance** of the two loops. Now

$$\Phi_2 = \int \vec{B_1} \cdot d\vec{a_2} = \int \left(\vec{\nabla} \times \vec{A_1}\right).d\vec{a_2} = \oint \vec{A_1}.d\vec{l_2}$$

Since $\vec{A_1} = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l_1}}{R} \Rightarrow \Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\vec{l_1}}{R} \right).d\vec{l_2}$

$$\Phi_2 = M_{21} I_1 \Rightarrow M_{21} = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l_1} \cdot d\vec{l_2}}{R}$$

This is the Neumann formula; it involves double line integral-one integration around loop1, the other around loop2.

Thus

(a) M_{21} is a purely geometrical quantity depends on sizes, shapes and relative position of two loops.

(b) $M_{21} = M_{12} = M$

If flux through loop 2, varies then induce emf in loop 2 is

$$\varepsilon_2 = -\frac{d\phi_2}{dt} = -M \frac{dI_1}{dt}.$$

Changing current not only induces an emf in any nearby loops, it also induces an emf in the source loop itself. Again field (and therefore flux) is proportional to the current.

$\Phi = LI$ where L is self inductance of the loop

If the current changes, the emf induced in the same loop is

$$\varepsilon = -L\frac{dI}{dt}.$$

Inductance is measured in **henries** (*H*); a henry is a volt-second per ampere.

Inductance (like capacitance) is an intrinsically positive quantity. Lenz's law, which is enforced by minus sign, which means the emf is in such a direction to oppose and change in current. For this reason, it is called a **back emf.** Whenever we try to alter the current, we must fight against this back emf.

4.1.3 Energy Stored in the field

It takes a certain amount of energy to start a current flowing in a circuit. The work done on a unit charge, against the back emf, in one trip around the circuit is $-\varepsilon$ (the mines sign is due to the fact that work is being done by us against the emf, not the work done by the emf). The amount of charge per unit time passing down the wire is *I*. So the total work done per unit time is, $\frac{dW}{dt} = -\varepsilon I = LI\frac{dI}{dt}$

If we start with zero current and build it up to a final value *I*,

The work done (Integrating the last equation over time) is $W = \frac{1}{2}LI^2$

Since $\Phi = LI = \int_S \vec{B}.d\vec{a} = \int_S \left(\vec{\nabla}\times\vec{A}\right).d\vec{a} = \oint_P \vec{A}.d\vec{l}$, where *P* is the perimeter of the loop and *S* is any surface bounded by *P*.

Therefore $W = \frac{1}{2\mu_0}I\oint \vec{A}.d\vec{l} \Rightarrow W = \frac{1}{2\mu_0}\oint\left(\vec{A}.\vec{I}\right)dl$

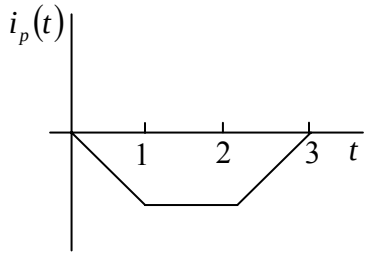
The generalization to volume current is: $W = \frac{1}{2}\int_V \left(\vec{A}.\vec{J}\right)d\tau$

We can simplify above equation as

$$W = \frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau$$

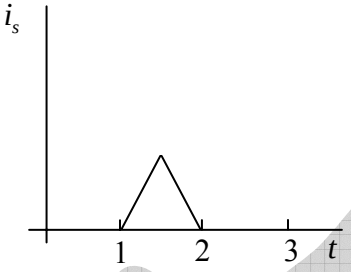
MCQ (Multiple Choice Questions)

Q1. A current i_p flows through the primary coil of a transformer. The graph of $i_p(t)$ as a function of time t is shown in the figure below.

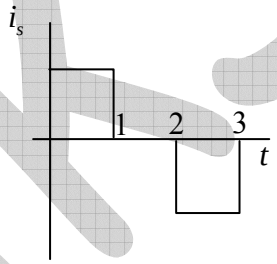


Which of the following graphs represents the current i_S in the secondary coil?

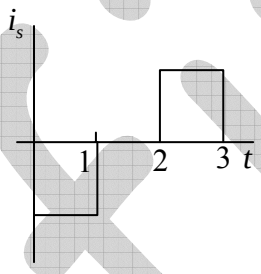
(a)



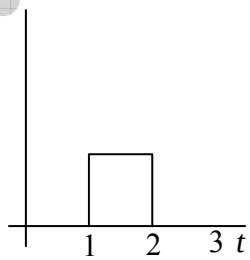
(b)



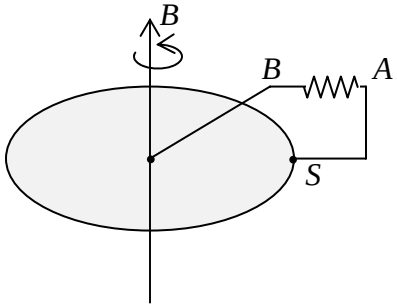
(c)



(d)



Q2. A horizontal metal disc rotates about the vertical axis in a uniform magnetic field pointing up as shown in the figure. A circuit is made by connecting one end A of a resistor to the centre of the disc and the other end B to its edge through a sliding contact. The current that flows through the resistor is



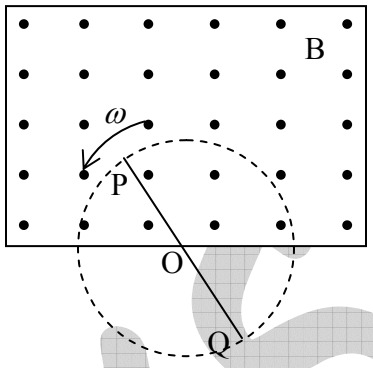
(a) zero

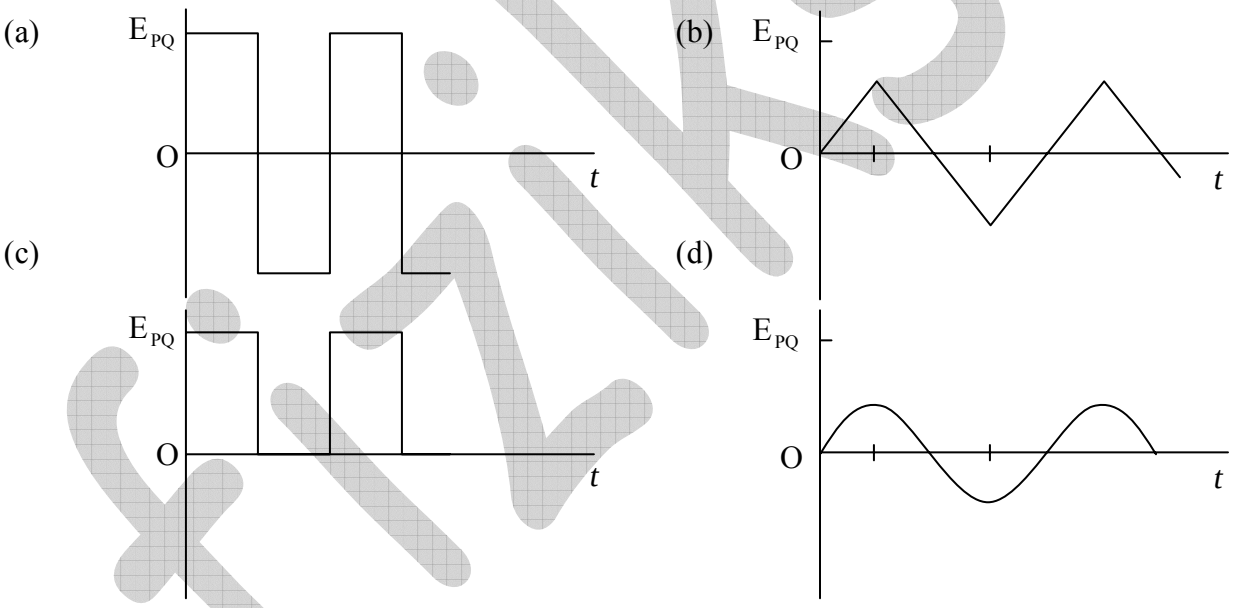
(b) DC from A to B

(c) DC from B to A

(d) AC

Q3. A uniform and constant magnetic field B coming out of the plane of the paper exists in a rectangular region as shown in the figure. A conducting rod PQ is rotated about O with a uniform angular speed ω in the plane of the paper. The emf E_{PQ} induced between P and Q is best represented by the graph





Q4. Consider a solenoid of radius R with n turns per unit length, in which a time dependent current $I = I_0 \sin \omega t$ (where $\omega R / c << 1$) flows. The magnitude of the electric field at a perpendicular distance $r > R$ from the axis of symmetry of the solenoid, is

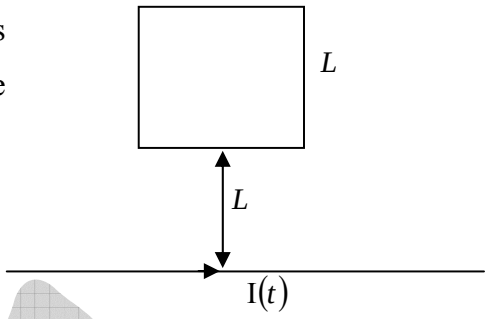
(a) 0

(b) $\frac{1}{2r}\omega\mu_0 n I_0 R^2 \cos \omega t$

(c) $\frac{1}{2}\omega\mu_0 n I_0 r \sin \omega t$

(d) $\frac{1}{2}\omega\mu_0 n I_0 r \cos \omega t$

Q5. An infinitely long wire carrying a current $I(t) = I_0 \cos \omega t$ is placed at a distance 'L' from a square loop of side 'L' as shown in figure. If the resistance of the loop is R, then the amplitude of the induced current in the loop is:



(a) $\frac{\mu_0}{4\pi} \cdot \frac{2LI_0\omega}{R}$ (b) $\frac{\mu_0}{4\pi} \cdot \frac{8LI_0\omega}{R} \ln 2$

(c) $\frac{\mu_0}{4\pi} \cdot \frac{4LI_0\omega}{R} \ln 2$ (d) $\frac{\mu_0}{4\pi} \cdot \frac{2LI_0\omega}{R} \ln 2$

Q6. A large circular coil of N turns and radius R carries a time varying current $I(t) = I_0 \sin \omega t$. A small circular coil of n turns and radius r $(r << R)$ is placed at the center of the large coil such that the coils are concentric and coplanar. The induced emf in the small coil

(a) Leads the current in the large coil by $\pi/2$

(b) Lags the current in the large coil by π

(c) is in phase with the current in the large coil

(d) Lags the current in the large coil by $\pi/2$

Q7. A thin conducting wire is bent into a circular loop of radius r and placed in a time dependent magnetic field of magnetic induction $\vec{B}(t) = B_0 e^{-\alpha t} \hat{z}$ *where* $B_0 > 0$, $\alpha > 0$ such that the plane of the loop is perpendicular to $\vec{B}(t)$. Then the induced emf in the loop is:

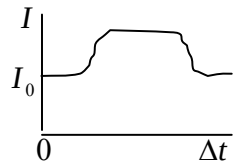
(a) $\pi r^2 \alpha B_0 e^{-\alpha t}$ (b) $\pi r^2 B_0 e^{-\alpha t}$

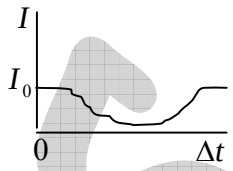
(c) $-\pi r^2 \alpha B_0 e^{-\alpha t}$ (d) $-\pi r^2 B_0 e^{-\alpha t}$

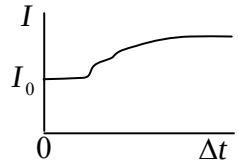
Q8. A square loop of wire, with sides of length L, lies in the first quadrant of the xy plane, with one corner at the origin. In this region there is a non-uniform time dependent magnetic field $\vec{B}(y,t) = Ky^3 t^2 \hat{z}$ (where K is a constant). Then the induced emf in the loop is

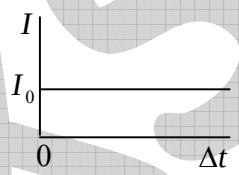
(a) $-\frac{1}{2} KtL^2$ (b) $-\frac{1}{4} KtL^4$ (c) $\frac{1}{2} KtL^5$ (d) $-\frac{1}{2} KtL^5$

Q9. A solenoid with an iron core is connected in series with a battery of emf V and it is found that a constant current I_0 passes through the solenoid. If at $t = 0$, the iron core is pulled out from the solenoid quickly in a time Δt, which one of the following could be a correct description of the current passing through the solenoid?

(a) 

(b) 

(c) 

(d) 

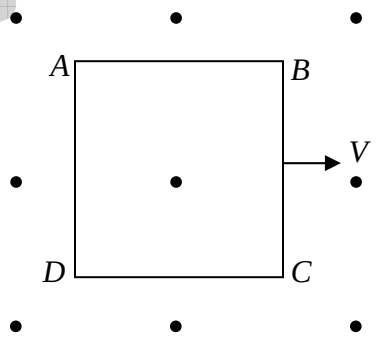
Q10. A metallic square loop *ABCD* is moving in its own plane with velocity V in a uniform magnetic field perpendicular to its plane as shown in the figure. An electric field is induced.

(a) in AD, but not in BC

(b) in BC, but not in AD

(c) neither in AD nor in BC

(d) in both AD and BC



NAT (Numerical Answer Type)

Q11. A small loop of wire of area $A = 0.01\ m^2$, $N = 40$ turns and resistance $R = 40\ \Omega$ is initially kept in a uniform magnetic field B in such a way that the field is normal to the loop. When it is pulled out of the magnetic field a total charge of $Q = 2\times10^{-5}\ C$ flows through the coil. The magnetic field B is……$\times10^{-3}$ *Tesla*

Q12. A conducting circular loop is placed in a uniform magnetic field of $0.02\,Tesla$, with its plane perpendicular to the field. If the radius of the loop starts shrinking at a constant rate $1.0\ mm/sec$, then the magnitude of e.m.f. induced in the loop, at the instant when the radius is $4.0\ cm$ will be…… μV. (Answer must be an integer)

Q13. A long solenoid, of radius *a*, is driven by alternating current, so that the field inside is sinusoidal: $\vec{B}(t) = B_0\cos(\omega t)\hat{z}$. A circular loop of wire, of radius $a/2$ and resistance *R*, is placed inside the solenoid, and coaxial with it. Then the amplitude of current induced in the loop, as a function of time is $\alpha\dfrac{\pi a^2 B_0\omega}{R}$. Then the value of α is……

MSQ (Multiple Select Questions)

Q14. Consider a solenoid of radius *R* with *n* turns per unit length, in which a time dependent current $I = I_0\sin\omega t$ $(\text{where } \omega R/c << 1)$ flows. Then which of the following statements are true for magnitude of the electric field at a perpendicular distance r from the axis of symmetry?

(a) Electric field $|\vec{E}| = 0$ for $r > R$

(b) Electric field $|\vec{E}| = \dfrac{1}{2r}\omega\mu_0 nI_0R^2\cos\omega t$ for $r > R$

(c) Electric field $|\vec{E}| = \dfrac{1}{2}\omega\mu_0 nI_0 r\sin\omega t$ for $r < R$

(d) Electric field $|\vec{E}| = \dfrac{1}{2}\omega\mu_0 nI_0 r\cos\omega t$ for $r < R$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (b)

Ans. 2: (b)

Ans. 3: (a)

Ans. 4: (b)

Solution: $\oint \vec{E}\cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t}\cdot d\vec{a}; \qquad \left(\vec{B} = \mu_0 nI(t)\hat{z}\right)$.

$$\Rightarrow \left|\vec{E}\right| \times 2\pi r = -\mu_0 n \frac{dI}{dt}\int_{r'=0}^{R} 2\pi r' dr' = -\mu_0 n \times I_0 \omega \cos\omega t \times \frac{2\pi R^2}{2}$$

$$\Rightarrow \left|\vec{E}\right| = -\frac{1}{2r} \times \omega \mu_0 n I_0 R^2 \cos\omega t$$

Ans. 5: (d)

Solution: Magnetic flux $\phi = \int \vec{B}\cdot d\vec{a} = \frac{\mu_0 I(t)}{2\pi}\int_{L}^{2L} \frac{1}{r}\hat{\phi}\cdot L dr\hat{\phi} \Rightarrow \phi(t) = \frac{\mu_0 I(t) L}{2\pi}\ln 2$

$$\Rightarrow \varepsilon = -\frac{d\phi}{dt} = \frac{\mu_0}{2\pi}\cdot\frac{LI_0\omega\sin\omega t}{R}\ln 2 \qquad \because I(t) = I_0\cos\omega t$$

$$\Rightarrow \varepsilon = \frac{\mu_0}{4\pi}\cdot\frac{2LI_0\omega\sin\omega t}{R}\ln 2$$

Ans. 6: (a)

Solution: $\varepsilon = -\frac{d\phi}{dt} = -A\frac{dB}{dt} \propto -\frac{dI}{dt} \propto -\cos\omega t \propto -\sin\left(\frac{\pi}{2}-\omega t\right) \propto \sin\left(\omega t - \frac{\pi}{2}\right)$

Ans. 7: (a)

Solution: $\varepsilon = -\frac{d\phi}{dt} = \pi r^2 \times \alpha B_0 e^{-\alpha t}$

Ans. 8: (d)

Solution: $\phi = \int B dx dy = Kt^2\int_0^L dx\int_0^L y^3 dy = \frac{Kt^2L^5}{4} \Rightarrow \varepsilon = -\frac{d\phi}{dt} = -\frac{1}{2}KtL^5$

Ans. 9: (a)

Ans. 10: (d)

NAT (Numerical Answer Type)

Ans. 11: **2**

Solution: Magnetic flux through the loop $\phi = NBA$

Induced e.m.f $\varepsilon = -\dfrac{d\phi}{dt}$ and induced current $i = -\dfrac{1}{R}\dfrac{d\phi}{dt} = \dfrac{dQ}{dt} \Rightarrow -\dfrac{1}{R}d\phi = dQ$.

$$\Rightarrow \frac{1}{40}\times\left(40\times B\times 0.01\right) = 2\times 10^{-5} \Rightarrow B = 2\times 10^{-3}\ T$$

Ans. 12: **5.0**

Solution: $\varepsilon = -\dfrac{d\phi}{dt} = -\dfrac{BdA}{dt} = -\dfrac{\pi B dr^2}{dr} = -\dfrac{2\pi B dr}{dt}r$

$$\varepsilon = 2\times\frac{22}{7}\times 2\times 10^{-2}\times\left(1\times 10^{-3}\right)\times 4\times 10^{-2} = 48\times 10^{-7}V = 4.8\times 10^{-6}V = 5.0\mu V$$

Ans. 13: **0.25**

Solution: Magnetic flux through the loop $\phi = \vec{B}.\vec{A} = B_0\cos\left(\omega t\right)\times\dfrac{\pi a^2}{4} = \dfrac{1}{4}\pi a^2 B_0\cos\left(\omega t\right)$

Induced emf $\varepsilon\left(t\right) = -\dfrac{d\phi}{dt} = \dfrac{1}{4}\pi a^2 B_0\omega\sin\omega t = \varepsilon_0\sin\omega t \Rightarrow \varepsilon_0 = \dfrac{1}{4}\pi a^2 B_0\omega$

Induced current $i\left(t\right) = \dfrac{\varepsilon\left(t\right)}{R} = \dfrac{\pi a^2 B_0\omega\sin\omega t}{4R} = i_0\sin\omega t \Rightarrow i_0 = \dfrac{\pi a^2 B_0\omega}{4R}$

MSQ (Multiple Select Questions)

Ans. 14: (b) and (d)

Solution: $\oint\vec{E}\cdot d\vec{l} = -\int\dfrac{\partial\overline{B}}{\partial t}\cdot d\overline{a};\qquad \left(\vec{B} = \mu_0 nI\left(t\right)\hat{z}\right).$

For $r > R \Rightarrow \left|\vec{E}\right|\times 2\pi r = -\mu_0 n\dfrac{dI}{dt}\int\limits_{r'=0}^{R} 2\pi r' dr' = -\mu_0 n\times I_0\omega\cos\omega t\times\dfrac{2\pi R^2}{2}$

$$\Rightarrow \left|\vec{E}\right| = -\frac{1}{2r}\times\omega\mu_0 nI_0 R^2\cos\omega t$$

For $r < R \Rightarrow \left|\vec{E}\right|\times 2\pi r = -\mu_0 n\dfrac{dI}{dt}\int\limits_{r'=0}^{r} 2\pi r' dr' = -\mu_0 n\times I_0\omega\cos\omega t\times\dfrac{2\pi r^2}{2}$

$$\Rightarrow \left|\vec{E}\right| = -\frac{1}{2}\times\omega\mu_0 nI_0 r\cos\omega t$$

Chapter - 5

Maxwell's Equations

5.1 Maxwell's Equation in Free Space

5.1.1 Electrodynamics before Maxwell's

(i) $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ (Gauss' Law),

(ii) $\vec{\nabla}\cdot\vec{B}=0$ (No name),

(iii) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ (Farday's Law),

(iv) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$ (Ampere's law).

Taking divergence of equation (iii) we get

$$\vec{\nabla}.(\vec{\nabla}\times\vec{E})=\vec{\nabla}.\left(-\frac{\partial\vec{B}}{\partial t}\right)=-\frac{\partial}{\partial t}\left(\vec{\nabla}.\vec{B}\right)=0.$$ So equation (iii) is valid.

Again taking divergence of equation (iv) we have $\vec{\nabla}.(\vec{\nabla}\times\vec{B})=\mu_0(\vec{\nabla}.\vec{J})$;

the left side must be zero, but the right side, in general, is not.

For steady currents $\vec{\nabla}.\vec{J}=0$, but for time varying fields ($\vec{\nabla}.\vec{J}=\frac{-\partial\rho}{\partial t}$) the Ampere's law can not be right.

5.1.2 How Maxwell fixed Ampere's Law

From continuity equation and Gauss Law

$$\vec{\nabla}.\vec{J}=\frac{-\partial\rho}{\partial t}=\frac{\partial}{\partial t}(\varepsilon_0\vec{\nabla}.\vec{E})=-\vec{\nabla}.\left(\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)\Rightarrow\vec{\nabla}.\left(\vec{J}+\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)=0.$$

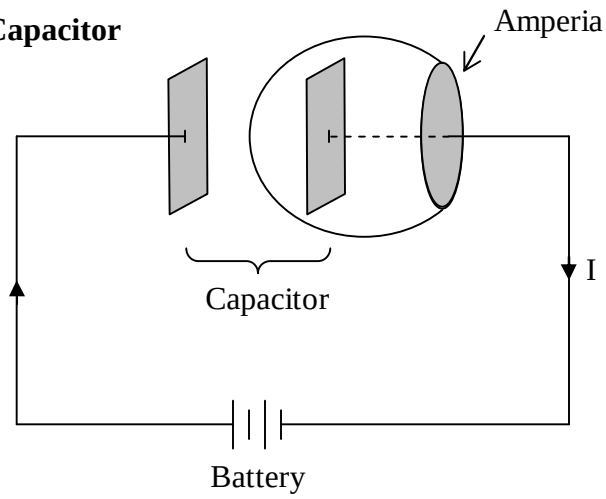
Thus $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

A changing electric field induces a magnetic field.

Maxwell called this extra term the **displacement current** $\vec{J}_d=\varepsilon_0\frac{\partial\vec{E}}{\partial t}$.

Integral form of Ampere's law $\oint\vec{B}.d\vec{l}=\mu_0 I_{enc}+\mu_0\varepsilon_0\int\left(\frac{\partial\vec{E}}{\partial t}\right).d\vec{a}$

5.1.3 Paradox of Charging Capacitor



Displacement current resolves the paradox of charging capacitor. If the capacitor plates are very close together, then the electric field between them is: $E = \frac{1}{\varepsilon_0}\sigma = \frac{1}{\varepsilon_0}\frac{Q}{A}$ where Q is the charge on the plate and A is its area.

Thus, between the plates $\frac{\partial E}{\partial t} = \frac{1}{\varepsilon_0 A}\frac{\partial Q}{\partial t} = \frac{1}{\varepsilon_0 A} I$

If we choose the flat surface, then $E = 0$ and $I_{enc} = I$

$$\Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right).d\vec{a} \Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 I \,.$$

If, on the other hand, we use the balloon-shaped surface, then $I_{enc} = 0$, then

$$\int \left(\frac{\partial \vec{E}}{\partial t}\right).d\vec{a} = \frac{I}{\varepsilon_0}$$

$$\Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 I_{enc} + \mu_0\varepsilon_0 \int \left(\frac{\partial \vec{E}}{\partial t}\right).d\vec{a} \Rightarrow \oint \vec{B}.d\vec{l} = \mu_0 I$$

So we get the same answer for either surface, though in the first case it comes from the genuine current and in the second from the displacement current.

5.1.4 Maxwell's Equation in Free Space

(i) $\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\varepsilon_0}$ (Gauss' Law),

(ii) $\vec{\nabla}\cdot\vec{B}=0$ (No name),

(iii) $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ (Farday's Law),

(iv) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (Ampere's law with Maxwell's correction).

5.2 Maxwell's Equation in Linear Isotropic Media

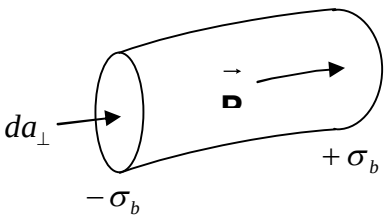
For materials that are subject to electric and magnetic polarization there is more convenient way to write Maxwell's equations. Inside polarized matter there will be accumulation of **"bound" charge** and **current** over which we don't have direct control. So we will reformulate Maxwell's equation in such a way as to make explicit reference only to those sources we control directly: the **"free" charges** and **currents**.

We know that an electric polarization $\vec{P}$ produces a bound charge density $\rho_b=-\vec{\nabla}.\vec{P}$.

Likewise, a magnetic polarization (or "magnetization") $\vec{M}$ results in a bound current $\vec{J}_b=\vec{\nabla}\times\vec{M}$.

Due to time varying field any change in the electric polarization involves a flow of (bound) charge, (call it polarization current $\vec{J}_P$), which must be included in the total current.

Consider a small piece of polarized material. The polarization introduces a charge density $\sigma_b=P$ at one end and $-\sigma_b$ at the other (since $\sigma_b=\vec{P}.\hat{n}$). If P now increases a bit, the charge on each end increases accordingly, giving a net current-carrying, $dI=\frac{\partial\sigma_b}{\partial t}da_\perp$



Thus the polarization current $\vec{J}_P=\frac{\partial\vec{P}}{\partial t}$

The polarization current has nothing to do with the bound current, $\vec{J}_b$. The bound current ($\vec{J}_b = \vec{\nabla} \times \vec{M}$) is associated with magnetization of the material and involves the spin and orbital motion of electrons.

In view of all this, the total charge density $\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla}.\vec{P}$,

and the total current density $\vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_P = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t}$.

Gauss's law can now be written as:

$$\vec{\nabla} \cdot \vec{\mathrm{E}} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla} \cdot \vec{\mathrm{E}} = \frac{1}{\varepsilon_0}\left(\rho_f - \vec{\nabla}.\vec{P}\right)$$

or $\vec{\nabla} \cdot \vec{D} = \rho_f$ where $\vec{D} = \varepsilon_0 \vec{E} + \vec{P}$

Now, Ampere's law (with Maxwell's term) becomes

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t},$$

or $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$ where $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

In terms of free charges and currents, then, Maxwell's equations read

(i) $\vec{\nabla} \cdot \vec{D} = \rho_f$, (ii) $\vec{\nabla} \cdot \vec{B} = 0$,

(iii) $\vec{\nabla} \times \vec{\mathrm{E}} = -\frac{\partial \vec{B}}{\partial t}$, (iii) $\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$

For linear media, $\vec{P} = \varepsilon_0 \chi_e \vec{E}$ and $\vec{M} = \chi_m \vec{H}$,

So $\vec{D} = \varepsilon \vec{E}$, and $\vec{H} = \frac{1}{\mu}\vec{B}$ where $\varepsilon = \varepsilon_0(1+\chi_e)$, $\mu = \mu_0(1+\chi_m)$

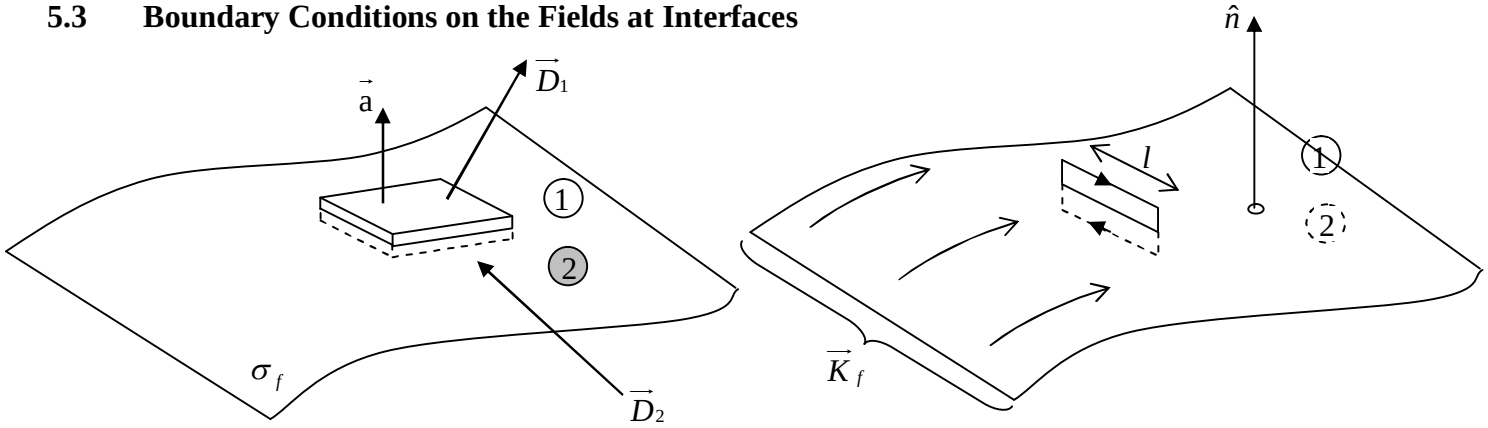
and displacement current $\vec{J}_d = \frac{\partial \vec{D}}{\partial t}$.

Integral form

$$\left.\begin{aligned}&\text{(i)}\ \oint_S \vec{D}\cdot d\vec{a} = Q_{f_{enc}}\\ &\text{(ii)}\ \oint_S \vec{B}\cdot d\vec{a} = 0\end{aligned}\right\}$$ over any closed surface S.

$$\left.\begin{aligned}&\text{(iii)}\ \oint_P \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int_s \vec{B}\cdot d\vec{a}\\ &\text{(iv)}\ \oint_P \vec{H}\cdot d\vec{l} = I_{f_{enc}} + \frac{d}{dt}\int_s \vec{D}\cdot d\vec{a}\end{aligned}\right\}$$ for any surface *S* bounded by the closed loop *P*.

5.3 Boundary Conditions on the Fields at Interfaces



$$\left.\begin{aligned}&\text{(a)}\ D^{\perp}_{above} - D^{\perp}_{below} = \sigma_f\\ &\text{(b)}\ B^{\perp}_{above} = B^{\perp}_{below}\end{aligned}\right\}$$ from (i) and (ii)

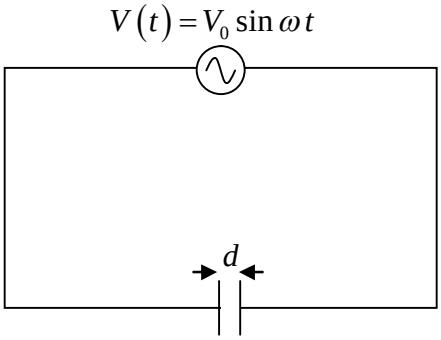
$$\left.\begin{aligned}&\text{(c)}\ \vec{E}^{\parallel}_{above} = \vec{E}^{\parallel}_{below}\\ &\text{(d)}\ \vec{H}^{\parallel}_{above} - \vec{H}^{\parallel}_{below} = \vec{K}_f \times \hat{n}\end{aligned}\right\}$$ from (iii) and (iv)

In particular, if there is no free charge or free current at the interface between medium1 and medium 2, then

$$\left.\begin{aligned}&\text{(a)}\ \varepsilon_1 E_1^{\perp} - \varepsilon_2 E_2^{\perp} = 0\\ &\text{(b)}\ B_1^{\perp} = B_2^{\perp}\end{aligned}\right\}$$ and $$\left.\begin{aligned}&\text{(c)}\ \vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}\\ &\text{(d)}\ \frac{1}{\mu_1}\vec{B}_1^{\parallel} - \frac{1}{\mu_2}\vec{B}_2^{\parallel} = 0\end{aligned}\right\}.$$

MCQ (Multiple Choice Questions)

Q1. An oscillating voltage $V(t) = V_0 \sin \omega t$ is applied across a parallel plate capacitor having a plate separation *d*. The displacement current density through the capacitor is



(a) $\frac{\varepsilon_0 \omega V_0 \cos \omega t}{d}$ (b) $\frac{\varepsilon_0 \mu_0 \omega V_0 \cos \omega t}{d}$

(c) $-\frac{\varepsilon_0 \mu_0 \omega V_0 \cos \omega t}{d}$ (d) $-\frac{\varepsilon_0 \omega V_0 \sin \omega t}{d}$

Q2. For a parallel plate capacitor of area *A* and separation *d*, across which a sinusoidal voltage $V_0 \sin \omega t$ is applied, the average value of the displacement current is (the capacitor is placed in vacuum)

(a) zero (b) $\frac{\varepsilon_0 A V_0 \omega}{d}$ (c) $\frac{\varepsilon_0 A \omega}{\sqrt{2} d}$ (d) $\frac{\varepsilon_0 A V_0 \omega}{\sqrt{2} d}$

Q3. For a parallel plate capacitor of area *A* and separation *d*, across which a sinusoidal voltage $V_0 \sin \omega t$ is applied, the peak value of the displacement current is (the capacitor is placed in vacuum)

(a) zero (b) $\frac{\varepsilon_0 A V_0 \omega}{d}$ (c) $\frac{\varepsilon_0 A \omega}{\sqrt{2} d}$ (d) $\frac{\varepsilon_0 A V_0 \omega}{\sqrt{2} d}$

Q4. For a parallel plate capacitor of area *A* and separation *d*, across which a sinusoidal voltage $V_0 \sin \omega t$ is applied, the r.m.s value of the displacement current is (the capacitor is placed in vacuum)

(a) zero (b) $\frac{\varepsilon_0 A V_0 \omega}{d}$ (c) $\frac{\varepsilon_0 A \omega}{\sqrt{2} d}$ (d) $\frac{\varepsilon_0 A V_0 \omega}{\sqrt{2} d}$

Q5. A parallel plate capacitor has circular plates of radius R. It is being charged by a current I. Then the magnetic induction $\vec{B}$ at a point between the plates at a distance $R/2$ from the axis of the capacitor is:

(a) $\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\phi}$ (b) $\vec{B} = \frac{\mu_0 I}{4\pi R}\hat{\phi}$ (c) $\vec{B} = \frac{\mu_0 I}{6\pi R}\hat{\phi}$ (d) $\vec{B} = \frac{\mu_0 I}{8\pi R}\hat{\phi}$

Q6. The divergence of a magnetic field $\vec{B}\left(\vec{r},t\right)$ from a time varying current density $\vec{J}\left(\vec{r},t\right)$ is

(a) always zero as there are no magnetic monopoles

(b) non-zero and proportional to the rate of change of electric field $\vec{E}\left(\vec{r},t\right)$ from the current density

(c) non-zero and proportional to the divergence of electric field $\vec{E}\left(\vec{r},t\right)$ from the current density

(d) non-zero and proportional to the current density

Q7. A charged capacitor (C) is connected in series with an inductor (L). When the displacement current reduces to zero, the energy of the LC circuit is

(a) stored entirely in its magnetic field.

(b) stored entirely in its electric field.

(c) distributed equally among its electric and magnetic fields.

(d) radiated out of the circuit.

Q8. Which one of the following Maxwell's equations implies the absence of magnetic monopoles?

(a) $\vec{\nabla}\cdot\vec{\mathrm{E}} = \rho/\varepsilon_0$ (b) $\vec{\nabla}\cdot\vec{\mathrm{B}} = 0$

(c) $\vec{\nabla}\times\vec{\mathrm{E}} = -\partial\vec{\mathrm{B}}/\partial t$ (d) $\vec{\nabla}\times\vec{\mathrm{B}} = (1/c^2)\partial\vec{\mathrm{B}}/\partial t + \mu_0\vec{J}$

Q9. Which of the following expressions is correct for a dielectric?

(a) $\oint_{surface}\vec{D}\cdot d\vec{a} = \left(Q_{free}\right)_{enclosed}$ (b) $\oint_{surface}\vec{\mathrm{E}}\cdot d\vec{a} = \frac{1}{\varepsilon}\left(Q_{free}\right)_{enclosed}$

(c) $\oint_{surface}\vec{D}\cdot d\vec{a} = \frac{1}{\varepsilon}\left(Q_{free}\right)_{enclosed}$ (d) $\oint_{surface}\vec{E}\cdot d\vec{a} = \frac{1}{\varepsilon}\left(Q_{free}\right)_{enclosed}$

Q10. Faraday Lenz law relates the rate of change of magnetic flux with the emf developed. Which of the following equations represents the above law?

(a) $\vec{\nabla}\times\vec{\mathrm{E}}=-\dfrac{\partial\vec{B}}{\partial t}$

(b) $\oint\vec{B}\cdot\vec{dl}=\varepsilon_0\mu_0\dfrac{\partial}{\partial t}\int\vec{\mathrm{E}}\cdot\vec{dS}$

(c) $\vec{\nabla}\times\vec{B}=\varepsilon_0\mu_0\dfrac{\partial\vec{\mathrm{E}}}{\partial t}$

(d) $\dfrac{d\vec{p}}{\partial t}=q\left(\vec{\mathrm{E}}+\vec{v}\times\vec{B}\right)$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (a)

Solution: Displacement current density $J_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V(t)}{\partial t} = \frac{\varepsilon_0 \omega V_0 \cos \omega t}{d}$

Ans. 2: (a)

Solution: $J_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t} = \frac{\varepsilon_0}{d} V_0 \omega \cos \omega t \Rightarrow I_d = J_d A = \frac{\varepsilon_0 A V_0 \omega}{d} \cos \omega t$

Average value $= 0$

Ans. 3: (b)

Solution: $J_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t} = \frac{\varepsilon_0}{d} V_0 \omega \cos \omega t \Rightarrow I_d = J_d A = \frac{\varepsilon_0 A V_0 \omega}{d} \cos \omega t$

Peak value $V_m = \frac{\varepsilon_0 A V_0 \omega}{d}$

Ans. 4: (d)

Solution: $J_d = \varepsilon_0 \frac{\partial E}{\partial t} = \frac{\varepsilon_0}{d}\frac{\partial V}{\partial t} = \frac{\varepsilon_0}{d} V_0 \omega \cos \omega t \Rightarrow I_d = J_d A = \frac{\varepsilon_0 A V_0 \omega}{d} \cos \omega t$

Peak value $V_m = \frac{\varepsilon_0 A V_0 \omega}{d}$ and R.M.S value $V_{rms} = \frac{\varepsilon_0 A V_0 \omega}{\sqrt{2} d}$

Ans. 5: (b)

Solution: $\oint \vec{B}.d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \int_S \frac{\partial \vec{E}}{\partial t}.d\vec{a}$

Consider an amperian loop of radius $r(r < R)$, then $I_{enc} = 0$ and since

$$E = \frac{Q(t)}{\pi R^2 \varepsilon_0} \Rightarrow \frac{\partial E}{\partial t} = \frac{I}{\pi R^2 \varepsilon_0}$$

Thus $\left|\vec{B}\right| \times 2\pi r = \mu_0 \varepsilon_0 \frac{I}{\pi R^2 \varepsilon_0} \times \pi r^2 \Rightarrow B(r) = \frac{\mu_0 I r}{2\pi R^2} \Rightarrow B(R/2) = \frac{\mu_0 I}{4\pi R} \Rightarrow \vec{B} = \frac{\mu_0 I}{4\pi R}\hat{\phi}$

Ans. 6: (a)

Ans. 7: (b)

Ans. 8: (b)

Ans. 9: (a)

Ans. 10: (a)

Chapter - 6

Electromagnetic Waves

6.1 Poynting Theorem ("Work Energy Theorem of Electrodynamics")

The work necessary to assemble a static charge distribution is

$$W_e = \frac{\varepsilon_0}{2}\int E^2 d\tau \text{ , where } E \text{ is the resulting electric field}$$

The work required to get currents going (against the back emf) is

$$W_m = \frac{1}{2\mu_0}\int B^2 d\tau \text{ , where } B \text{ is the resulting magnetic field}$$

This suggests that the total energy in the electromagnetic field is

$$U_{em} = \frac{1}{2}\int\left(\varepsilon_0 E^2 + \frac{B^2}{\mu_0}\right)d\tau .$$

Suppose we have some charge and current configuration which at time t, produces fields $\vec{E}$ & $\vec{B}$. In next instant dt the charges moves around a bit. The work is done by electromagnetic forces acting on these charges in the interval dt.

According to Lorentz Force Law, the work done on a charge 'q' is

$$\vec{F}\cdot d\vec{l} = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}\,dt = q\vec{E}\cdot\vec{v}\,dt .$$

Now $q = \rho d\tau$ and $\rho\vec{v} = \vec{J}$, so the rate at which work is done on all the charges in a volume V is

$$\frac{dW}{dt} = \int_V \left(\vec{E}.\vec{J}\right)d\tau .$$

$\vec{E}\cdot\vec{J}$ is the work done per unit time, per unit volume- which is the power delivered per unit volume. Use Ampere's–Maxwell law to eliminate $\vec{J}$:

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) = \mu_0\left(\vec{E}\cdot\vec{J}\right) + \mu_0\varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

$$\Rightarrow \vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot\left(\vec{\nabla}\times\vec{B}\right) - \varepsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$$

Since $\vec{\nabla}.\left(\vec{E}\times\vec{B}\right)=\vec{B}.\left(\vec{\nabla}\times\vec{E}\right)-\vec{E}.\left(\vec{\nabla}\times\vec{B}\right)$ *and* $\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$

It follows that $\vec{E}.\left(\vec{\nabla}\times\vec{B}\right)=-\vec{B}.\frac{\partial\vec{B}}{\partial t}-\vec{\nabla}.\left(\vec{E}\times\vec{B}\right)$

$$\Rightarrow \vec{E}\cdot\vec{J}=-\frac{1}{2}\frac{\partial}{\partial t}\left(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2\right)-\frac{1}{\mu_0}\vec{\nabla}.\left(\vec{E}\times\vec{B}\right)$$

Then $$\frac{dW}{dt}=-\frac{d}{dt}\int_V\frac{1}{2}\left(\varepsilon_0 E^2+\frac{1}{\mu_0}B^2\right)d\tau-\frac{1}{\mu_0}\oint_S\left(\vec{E}\times\vec{B}\right)\cdot d\vec{a},$$

where S is the surface bounding V.

This is **Poynting's theorem**; it is the "work energy theorem" of electrodynamics.

The ***first integral*** on the right is the total energy stored in the fields, U_{em}.

The ***second term*** evidently, represents the rate at which energy is carries out of V, across its boundary surface, by the electromagnetic fields.

Poynting's theorem says, that, "the work done on the charges by the electromagnetic force is equal to the decrease in energy stored in the field, less the energy that flowed out through the surface".

The energy per unit time, per unit area, transported by the fields is called the ***Poynting vector***

$$\vec{S}\equiv\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right).$$

$\vec{S}\cdot d\vec{a}$ is the energy per unit time crossing the infinitesimal surface $d\vec{a}$ - the energy or energy flux density.

6.2 Waves in One Dimension (Sinusoidal waves)

6.2.1 The Wave Equation

A wave propagating with speed v in z-direction can be expressed as:

$$\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

It admits as solutions all functions of the form

$$f\left(z,t\right)=g\left(z-vt\right).$$

But functions of the form $g(z-vt)$ are not the only solutions. The wave equation involves the square of v, so we can generate another class of solutions by simply changing the sign of the velocity:

$$f(z,t)=h(z+vt).$$

This, of course, represents a wave propagating in the *negative* z-direction.

The *most general* solution to the wave equation is the sum of a wave to the right and a wave to the left:

$$f(z,t)=g(z-vt)+h(z+vt).$$

(Notice that the wave equation is **linear**: the sum of any two solutions is itself a solution.) *Every* solution to the wave equation can be expressed in this form.

Example: Check which of the following functions satisfy the wave equation (where symbols have their usual meaning and A,α are constants of suitable dimensions)

(a) $f(z,t)=Ae^{-\alpha(z-vt)^2}$ (b) $f(z,t)=A\sin[\alpha(z-vt)]$

(c) $f(z,t)=\dfrac{A}{\alpha(z-vt)^2+1}$ (d) $f(z,t)=Ae^{-\alpha(z^2+vt)}$

(e) $f(z,t)=A\sin(\alpha z)\cos(\alpha vt)^3$

Solution: (a) $f(z,t)=Ae^{-\alpha(z-vt)^2}\Rightarrow\dfrac{\partial f}{\partial z}=Ae^{-\alpha(z-vt)^2}\times-2\alpha(z-vt)$

$$\frac{\partial^2 f}{\partial z^2}=-2\alpha Ae^{-\alpha(z-vt)^2}+\left(-2\alpha(z-vt)\right)Ae^{-\alpha(z-vt)^2}\times-2\alpha(z-vt)$$

$$\frac{\partial^2 f}{\partial z^2}=Ae^{-\alpha(z-vt)^2}\left[-2\alpha+4\alpha^2(z-vt)^2\right]$$

$$f(z,t)=Ae^{-\alpha(z-vt)^2}\Rightarrow\frac{\partial f}{\partial t}=Ae^{-\alpha(z-vt)^2}\times2\alpha v(z-vt)$$

$$\Rightarrow\frac{\partial^2 f}{\partial t^2}=-2\alpha v^2Ae^{-\alpha(z-vt)^2}+2\alpha v(z-vt)\times Ae^{-\alpha(z-vt)^2}\times2\alpha v(z-vt)$$

$$\Rightarrow\frac{\partial^2 f}{\partial t^2}=v^2Ae^{-\alpha(z-vt)^2}\left[-2\alpha+4\alpha^2(z-vt)^2\right]$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}\text{ (Wave Equation)}$$

(b) $f(z,t)=A\sin[\alpha(z-vt)] \Rightarrow \dfrac{\partial^2 f}{\partial z^2}=\dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (Wave Equation)

(c) $f(z,t)=\dfrac{A}{\alpha(z-vt)^2+1} \Rightarrow \dfrac{\partial^2 f}{\partial z^2}=\dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (Wave Equation)

(d) $f(z,t)=Ae^{-\alpha(z^2+vt)} \Rightarrow \dfrac{\partial^2 f}{\partial z^2}\neq\dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (Not a Wave Equation)

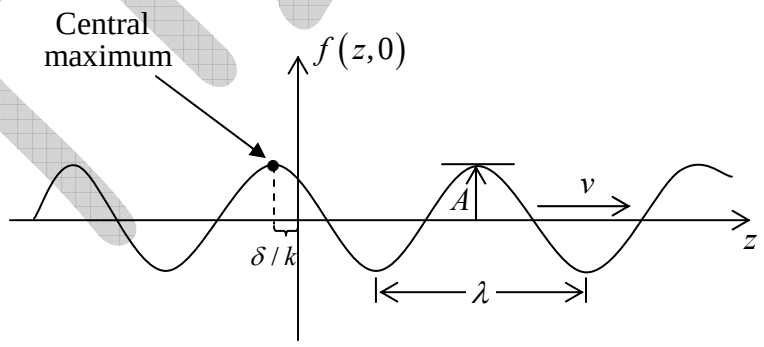
(e) $f(z,t)=A\sin(\alpha z)\cos(\alpha vt)^3 \Rightarrow \dfrac{\partial^2 f}{\partial z^2}=\dfrac{1}{v^2}\dfrac{\partial^2 f}{\partial t^2}$ (Wave Equation)

6.2.2 Terminology

Let us consider a function $f(z,t)=A\cos[k(z-vt)+\delta]$, where A is the ***amplitude*** of the wave (it is positive, and represents the maximum displacement from equilibrium).

The argument of the cosine is called the ***phase,*** and δ is the ***phase constant*** (normally, we use a value in the range $0\le\delta<2\pi$).

Figure given below shows this function at time $t=0$. Notice that at $z=vt-\delta/k$, the phase is zero; let's call this the "central maximum." If $\delta=0$, central maximum passes the origin at time $t=0$; more generally δ/k is the distance by which the central maximum (and therefore the entire wave) is "***delayed***."



Finally k is the ***wave number***; it is related to the ***wavelength*** λ as $\lambda=\dfrac{2\pi}{k}$, for when z advances by $\dfrac{2\pi}{k}$, the cosine executes one complete cycle.

As time passes, the entire wave train ***proceeds to the right***, at speed v. ***Time period*** of one complete cycle is $T = \frac{2\pi}{kv}$.

The ***frequency*** ν (number of oscillations per unit time) is $\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$.

The ***angular frequency*** $\omega = 2\pi\nu = kv$

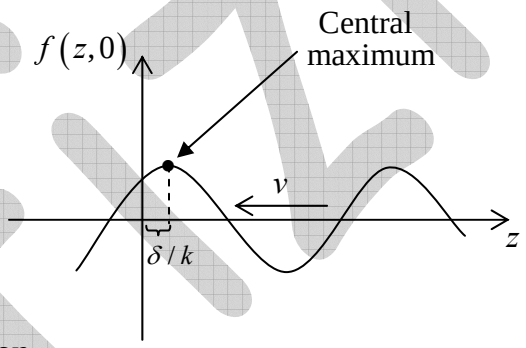
In terms of angular frequency ω, the sinusoidal wave can be represented as

$$f(z,t) = A\cos(kz - \omega t + \delta).$$

A sinusoidal oscillation of wave number k and angular frequency ω ***traveling to the left*** would be written

$$f(z,t) = A\cos(kz + \omega t - \delta).$$

Comparing this with the wave traveling to the right reveals that, in effect, we could *simply switch the sign of k* to produce a wave with the same amplitude, phase constant, frequency, and wavelength, traveling in the opposite direction.



6.2.3 Complex notation

In view of ***Euler's formula*** $e^{i\theta} = \cos\theta + i\sin\theta$,

the sinusoidal wave $f(z,t) = A\cos(kz - \omega t + \delta)$ can be written as

$$f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right],$$

where $\mathrm{Re}(\eta)$ denotes the real part of the complex number η. This invites us to introduce the ***complex wave function***

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$$

with the ***complex amplitude*** $\tilde{A} = Ae^{i\delta}$ absorbing the phase constant.

The actual wave function is the real part of $\tilde{f}$:

$$f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right].$$

The advantage of the complex notation is that exponentials are much easier to manipulate than sines and cosines.

6.2.4 Polarization

In ***longitudinal*** wave, the displacement from the equilibrium is along the direction of propagation. Sound waves, which are nothing but compression waves in air, are longitudinal.

Electromagnetic waves are ***transverse*** in nature. In a transverse wave displacement is perpendicular to the direction of propagation.

There are two dimensions perpendicular to any given line of propagation. Accordingly, transverse waves occur in two independent state of polarization:

"Vertical" polarization $\tilde{f}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{x}$,

"Horizontal" polarization $\tilde{f}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{y}$,

or along any other direction in the *xy* plane

$$\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}.$$

The ***polarization vector*** $\hat{n}$ defines the plane of vibration. Because the waves are transverse, $\hat{n}$ is perpendicular to the direction of propagation:

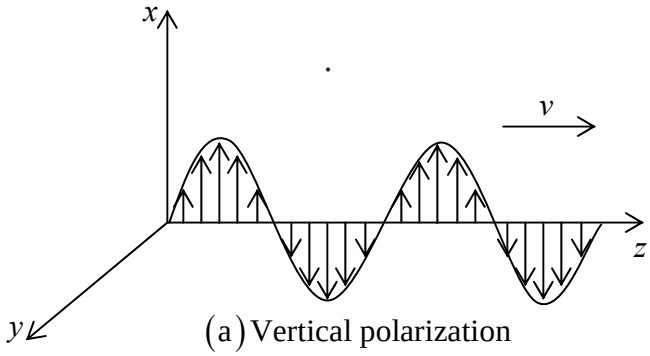
$$\hat{n}.\hat{z} = 0$$

In terms of polarization angle θ ,

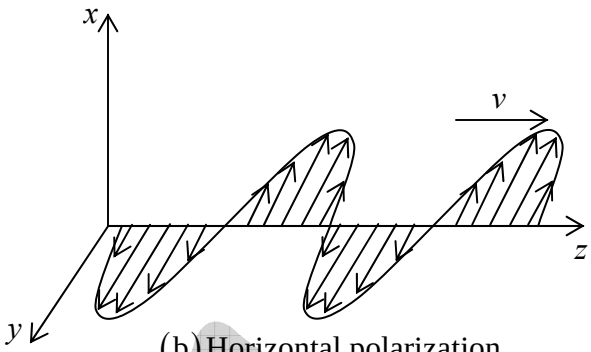
$$\hat{n} = \cos\theta\hat{x} + \sin\theta\hat{y}$$

Thus wave in figure(c) can be considered a superposition of two waves-one horizontally polarized, the other vertically:

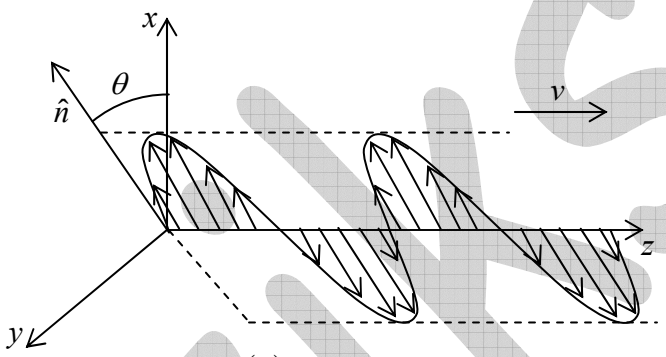
$$\tilde{f}(z,t) = \left(\tilde{A}\cos\theta\right)e^{i(kz-\omega t)}\hat{x} + \left(\tilde{A}\sin\theta\right)e^{i(kz-\omega t)}\hat{y}$$



(a) Vertical polarization



(b) Horizontal polarization



(c) Polarization vector

6.3 Electromagnetic Waves in Vacuum

6.3.1 The Wave Equation for $\vec{E}$ and $\vec{B}$

Maxwell's equations in free space ($\rho = 0$ *and* $\vec{J} = 0$) can be written as,

(i) $\vec{\nabla}.\vec{E} = 0$ (ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t}$ (iv) $\vec{\nabla}\times\vec{B} = \varepsilon_0\mu_0\dfrac{\partial\vec{E}}{\partial t}$

Taking curl of equation (iii) and using equation (i) & (ii) we get,

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) \Rightarrow \vec{\nabla}\left(\vec{\nabla}.\vec{E}\right) - \nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{B}\right) \Rightarrow -\nabla^2\vec{E} = -\frac{\partial}{\partial t}\left(\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\Rightarrow \nabla^2\vec{E} = \mu_0\varepsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Similarly, $\nabla^2\vec{B} = \mu_0\varepsilon_0\dfrac{\partial^2\vec{B}}{\partial t^2}$

Thus, $\vec{E}$ and $\vec{B}$ satisfy the wave equation $\nabla^2 f = \frac{1}{v}\frac{\partial^2 f}{\partial t^2}$.

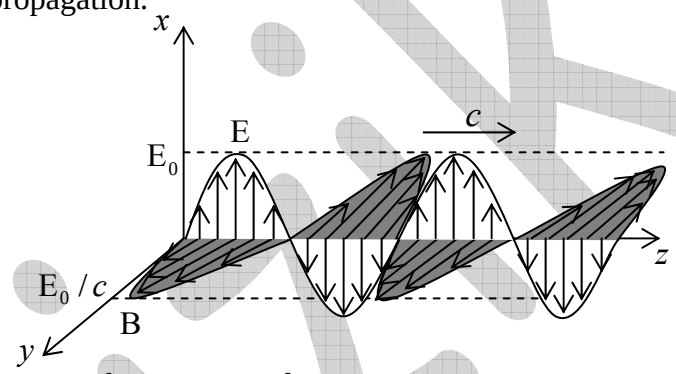
So, EM waves travels with a speed

$$v = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3\times10^8 m/s = c(\textit{velocity of light in free space})$$

where $\mu_0 = 4\pi\times10^{-7}\ {}^{N}\!/_{A^2}$, $\varepsilon_0 = 8.86\times10^{-12}\ {}^{C^2}\!/_{Nm^2}$

6.3.2 Monochromatic Plane Waves

Suppose waves are traveling in the *z*-direction and have no *x* or *y* dependence; these are called ***plane waves*** because the fields are uniform over every plane perpendicular to the direction of propagation.



The plane waves can be represented as:

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)},\ \tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

where $\tilde{E}_0$ and $\tilde{B}_0$ are the (complex) amplitudes (the physical fields, of course are the real parts of $\tilde{E}$ and $\tilde{B}$).

Since $\vec{\nabla}.\vec{E} = 0$ and $\vec{\nabla}.\vec{B}=0$, it follows that

$$\left(\tilde{E}_0\right)_z = \left(\tilde{B}_0\right)_z = 0$$

That is, electromagnetic waves are ***transverse***: the electric and magnetic fields are perpendicular to the direction of propagation.

Also $$\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow -k\left(\tilde{E}_0\right)_y = \omega\left(\tilde{B}_0\right)_x \quad , \quad \mathrm{k}\left(\tilde{E}_0\right)_x = \omega\left(\tilde{B}_0\right)_y$$

In compact form $\vec{\tilde{B}}_0 = \frac{k}{\omega}(\hat{z}\times\vec{\tilde{E}}_0)$

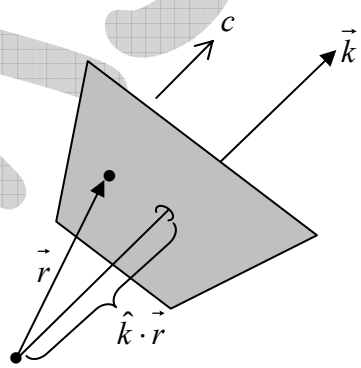
Evidently $\vec{E}$ & $\vec{\mathrm{B}}$ are in phase and mutually perpendicular; their (real) amplitudes are related by,

$$B_o = \frac{k}{\omega}E_o = \frac{1}{c}E_0 .$$

There is nothing special about the z direction; we can generalize the monochromatic plane waves traveling in an arbitrary direction. The ***propagation vector*** or ***wave vector*** $\vec{k}$ points in the direction of propagation, whose magnitude is the wave number *k*. The scalar product $\vec{k}.\vec{r}$ is the appropriate generalization of *kz*, so

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0 e^{i\left(\vec{k}.\vec{r}-\omega t\right)}\hat{n},$$

$$\vec{\tilde{B}}(\vec{r},t) = \frac{1}{c}\vec{\tilde{E}}_0 e^{i\left(\vec{k}.\vec{r}-\omega t\right)}\left(\hat{k}\times\hat{n}\right) = \frac{1}{c}\hat{k}\times\vec{\tilde{E}}$$



where $\hat{n}$ is polarization vector.

Also $\vec{\tilde{E}}(\vec{r},t) = -c\left(\hat{k}\times\vec{\tilde{B}}\right)$

Because $\vec{E}$ is transverse, ($\vec{B}$ is also transverse):

$$\hat{n}\cdot\hat{k} = 0 .$$

The actual (real) electric and magnetic fields in a monochromatic plane wave with propagation vector $\vec{k}$ and polarization $\hat{n}$ are

$$\vec{E}(\vec{r},t) = E_0\cos\left(\vec{k}.\vec{r}-\omega t+\delta\right)\hat{n}$$

$$\vec{B}(\vec{r},t) = \frac{1}{c}E_0\cos\left(\vec{k}.\vec{r}-\omega t+\delta\right)\left(\hat{k}\times\hat{n}\right)$$

Example: Write down the electric and magnetic fields for a plane monochromatic wave of amplitude E_0, frequency ω and phase angle zero that is

(a) Traveling in the *y*-direction and polarized in the *x*-direction.

(b) Traveling in the direction from the origin to the point $(1,1,1)$ with polarization parallel to *xy* plane.

Solution: (a) $\vec{E} = E_0 \cos(ky - \omega t)\hat{x}$ and

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E} = \frac{E_0}{c}\cos(ky - wt)(\hat{y} \times \hat{x}) = -\frac{E_0}{c}\cos(ky - wt)\hat{z}$$

(b) $\vec{k} = \frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right); \hat{n} = \alpha\hat{x} + \beta\hat{y} \Rightarrow \hat{n}.\hat{k} = 0 \Rightarrow \alpha = -\beta = \frac{1}{\sqrt{2}} \Rightarrow \hat{n} = \frac{\hat{x}-\hat{y}}{\sqrt{2}}$

$$\vec{E}(r,t) = E_0 \cos(\vec{k}.\vec{r} - \omega t)\hat{n} \text{ where } \vec{k}.\hat{r} = \frac{\omega}{\sqrt{3}c}(x+y+z)$$

$$\vec{E}(r,t) = E_0 \cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left(\frac{\hat{x}-\hat{y}}{2}\right)$$

$$\vec{B} = \frac{1}{\omega}\vec{k} \times \vec{E} = \frac{1}{\omega}E_0 \cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right]\left[\frac{\omega}{c}\left(\frac{\hat{x}+\hat{y}+\hat{z}}{\sqrt{3}}\right) \times \left(\frac{\hat{x}-\hat{y}}{2}\right)\right]$$

$$\vec{B} = \frac{E_0}{2\sqrt{3}c}\cos\left[\frac{\omega}{\sqrt{3}c}(x+y+z) - \omega t\right](\hat{x}+\hat{y}-2\hat{z})$$

6.3.3 Energy and Momentum in Electromagnetic Wave

The energy per unit volume stored in electromagnetic field is

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

In case of monochromatic plane wave

$$B^2 = \frac{E^2}{c^2} = \mu_0\varepsilon_0 E^2$$

So the electric and magnetic contributions are equal i.e. $u_E = u_B = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2\mu_0}B^2$.

$$u = u_E + u_B = \varepsilon_0 E^2 = \varepsilon_0 E_0^{\,2}\cos^2(kz - wt + \delta) \ .$$

As the wave travels, it carries this energy along with it. The energy flux density (energy per unit area, per unit time) transported by the fields is given by the ***Pointing vector***

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

For monochromatic plane wave propagating in the z-direction,

$$\vec{S} = c\varepsilon_0 E_0^2 \cos^2(kz - wt + \delta)\hat{z} = cu\hat{z}\,.$$

The energy per unit time, per unit area, transported by the wave is therefore uc.

Electromagnetic fields not only carry energy, they also carry ***momentum.*** The momentum density stored in the field is

$$\vec{\wp} = \frac{1}{c^2}\vec{S}\,.$$

For monochromatic plane wave,

$$\vec{\wp} = \frac{1}{c}\varepsilon_0 E_0{}^2 \cos^2(kz - wt + \delta)\hat{z} = \frac{1}{c}u\hat{z}\,.$$

Average energy density $\langle u \rangle = \frac{1}{2}\varepsilon_0 E_0^2$,

Average of Poynting vector $\langle \vec{S} \rangle = \frac{1}{2}c\varepsilon_0 E_0^2 \hat{z}$,

Average momentum density $\langle \vec{\wp} \rangle = \frac{1}{2c}\varepsilon_0 E_0^2 \hat{z}$.

The average power per unit area transported by an electromagnetic wave is called the ***intensity***

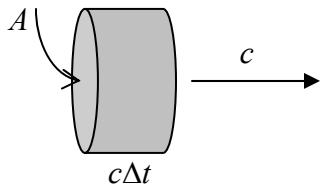
$$I = \langle S \rangle = \frac{1}{2}c\varepsilon_0 E_0^2\,.$$

Note:

(a) When light falls on ***perfect absorber*** it delivers its momentum to the surface. In a time Δt the momentum transfer is

$$\Delta p = \langle \wp \rangle Ac\Delta t\,,$$

so the ***radiation pressure*** (average force per unit area) is



$$P = \frac{1}{A}\frac{\Delta p}{\Delta t} = \frac{1}{2}\varepsilon_0 E_0^2 = \frac{I}{c}.$$

(b) When light falls on ***perfect reflector***, the ***radiation pressure***

$$P = \frac{2I}{c}$$

because the momentum changes direction, instead of being absorbed.

Example: The electric and magnetic fields of an electromagnetic waves in the free space are $\vec{\mathrm{E}} = \mathrm{A}_0 \cos(\omega t - kz)\hat{x}$, $\vec{\mathrm{B}} = \mathrm{B}_0 \cos(\omega t - kz)\hat{y}$ where $\hat{x}, \hat{y}$ being unit vectors in x and y directions respectively. Then find the intensity of electromagnetic wave.

Solution: $\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) = \frac{A_0 B_0}{\mu_0}\cos^2(\omega t - kz)\hat{z} \Rightarrow I = \langle \vec{S} \rangle = \frac{1}{2}.\frac{A_0 B_0}{\mu_0}$

6.4 Electromagnetic Waves in Matter

Inside matter, but in regions where there is no free charge or free current (ρ_f *and* $\vec{J}_f$)

Maxwell's Equation becomes,

(i) $\vec{\nabla}.\vec{D} = 0$ (ii) $\vec{\nabla}.\vec{B} = 0$

(iii) $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (iv) $\vec{\nabla}\times\vec{H} = \frac{\partial \vec{D}}{\partial t}$

If the medium is linear and homogeneous, $\vec{D} = \varepsilon\vec{E}$ and $\vec{\mathrm{H}} = \frac{1}{\mu}\vec{B}$

Now the wave equation inside matter is

$$\nabla^2\vec{E} = \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2} \quad and \quad \nabla^2\vec{B} = \mu\varepsilon\frac{\partial^2\vec{B}}{\partial t^2}.$$

Thus EM waves propagate through a linear homogenous medium at a speed

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{n} \text{ where } n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

Thus $n = \sqrt{\varepsilon_r}$ is the **index of refraction** (since $\mu_r = 1$ for non-magnetic material).

The energy density $u = \frac{1}{2}\left(\varepsilon E^2 + \frac{1}{\mu}B^2\right) \Rightarrow \langle u \rangle = \frac{1}{2}\varepsilon E_0^2$

The Poynting vector $\vec{S} = \frac{1}{\mu}\left(\vec{E} \times \vec{B}\right) \Rightarrow \langle \vec{S} \rangle = \frac{1}{2} v \varepsilon E_0^2 \hat{z}$

Intensity $I = \langle \vec{S} \rangle = \frac{1}{2} \varepsilon v E_0^2$

Thus in a medium $c \to v$, $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$

6.5 Electromagnetic Waves in Conductors

Any initial free charge density $\rho_f(0)$ given to conductor dissipate in a characteristic time $\tau \equiv \varepsilon / \sigma$ where σ is conductivity and

$$\rho_f(t) = e^{-(\sigma/\varepsilon)t} \rho_f(0)$$

This reflects the familiar fact that if we put some free charge on conductor, it will flow out to the edges.

Free current density in a conductor is $\vec{J}_f = \sigma \vec{E}$.

Thus Maxwell's equations inside conductor are

(i) $\vec{\nabla} \cdot \vec{E} = 0$ (ii) $\vec{\nabla} \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (iv) $\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial \vec{E}}{\partial t} + \mu\sigma \vec{E}$

We get modified wave equation for $\vec{E}$ *and* $\vec{B}$ as

$$\nabla^2 \vec{E} = \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma \frac{\partial \vec{E}}{\partial t} \quad \text{and} \quad \nabla^2 \vec{B} = \mu\varepsilon \frac{\partial^2 \vec{B}}{\partial t} + \mu\sigma \frac{\partial \vec{B}}{\partial t}$$

The admissible plane wave solution is

$\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 e^{i(\tilde{k}z - \omega t)}$, $\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_0 e^{i(\tilde{k}z - \omega t)}$ where "wave number" $\tilde{k}$ is complex

If we put the solution in wave equation, we get $\tilde{k}^2 = \mu\varepsilon\omega^2 + i\mu\sigma\omega$.

Let $\tilde{k} = k + i\kappa$ where k and κ are real and imaginary part of $\tilde{k}$.

$$\Rightarrow \quad k = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \quad \text{and} \quad \kappa = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

Thus, $\vec{\tilde{E}}(z,t) = \vec{\tilde{E}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$, $\vec{\tilde{B}}(z,t) = \vec{\tilde{B}}_0 e^{-\kappa z} e^{i(kz - \omega t)}$

The distance it takes to reduce the amplitude by a factor of $\frac{1}{e}$ is called the ***skin depth* (*d*)**

$$d = \frac{1}{\kappa};$$

it is a measure of how far the wave penetrates into the conductor.

The real part of $\tilde{k}$ determines the wavelength, the propagation speed, and the index of refraction:

$$\lambda = \frac{2\pi}{k},\ v = \frac{\omega}{k},\quad n = \frac{ck}{\omega}$$

Like any complex number, $\tilde{k}$ can be expressed in terms of its modulus and phase:

$$\tilde{k} = Ke^{i\phi}$$

where $K = \left|\tilde{k}\right| = \sqrt{k^2 + \kappa^2} = \omega\sqrt{\varepsilon\mu}\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}$ and $\phi \equiv \tan^{-1}\left(\kappa/k\right)$

The complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ *and* $\tilde{B}_0 = B_0 e^{i\delta_B}$ are related by

$$\tilde{B}_0 = \frac{\tilde{k}}{\omega}\tilde{E}_0 \Rightarrow B_0 e^{i\delta_B} = \frac{Ke^{i\phi}}{\omega}E_0 e^{i\delta_E}.$$

Evidently the electric and magnetic fields are no longer in phase; in fact $\delta_B - \delta_E = \phi$, the *magnetic field* ***lags behind*** *the electric fields.*

$$\frac{B_0}{E_0} = \frac{K}{\omega} = \left[\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}\right]^{1/2}$$

Thus, $\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E)\hat{x}$, $\vec{B}(z,t) = B_0 e^{-\kappa z}\cos(kz - \omega t + \delta_E + \phi)\hat{y}$

Note:

(a) In a poor conductor ($\sigma << \omega\varepsilon$)

$$\kappa = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$ i.e. independent of frequency.

(b) In a very good conductor ($\sigma >> \omega\varepsilon$)

$$\kappa = k = \sqrt{\frac{\omega\mu\sigma}{2}} \Rightarrow d = \frac{1}{\kappa} = \sqrt{\frac{2}{\omega\mu\sigma}} = \sqrt{\frac{1}{\pi f \mu\sigma}}$$

(c) When an electromagnetic wave strikes a perfect conductor ($\sigma \to \infty$) then all waves are reflected back i.e. $\tilde{E}_{0R} = -\tilde{E}_{0I}$ and $\tilde{E}_{0T} = 0$.

Example: An electromagnetic wave of frequency $10\,GHz$ is propagating through a conductor having conductivity $6\times10^{7}\,(\Omega m)^{-1}$ and $\sigma >> \omega\varepsilon$. Then find the skin depth of the conductor.

Solution: $d = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{1}{\pi f\sigma\mu}} = \sqrt{\frac{1}{3.14\times10^{10}\times6\times10^{7}\times4\pi\times10^{-7}}} = 6\times10^{-7}\,m$

Example: An electromagnetic plane wave is propagating inside a conductor with electric field $\vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{x}$. Then calculate the intensity of the wave inside the conductor.

Solution: $\because \vec{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\hat{x} \Rightarrow \vec{B} = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\hat{y}$

$$\vec{S} = \frac{1}{\mu}(\vec{E}\times\vec{B}) = \frac{1}{\mu}E_0B_0e^{-2\kappa z}\cos(kz-\omega t+\delta_E)\cos(kz-\omega t+\delta_E+\phi)$$

$$I = \langle\vec{S}\rangle = \frac{1}{\mu}E_0B_0e^{-2\kappa z}\times\frac{1}{2\pi}\int\cos\theta\cos(\theta+\phi)\,d\theta = \frac{1}{2\mu}E_0B_0e^{-2\kappa z}\cos\phi$$

$$\tan\phi = \frac{\kappa}{k} \Rightarrow \cos\phi = \frac{k}{\sqrt{k^2+\kappa^2}} \text{ and } B_0 = E_0\times\frac{\sqrt{k^2+\kappa^2}}{\omega}$$

$$\Rightarrow I = \frac{k}{2\mu\omega}E_0^2e^{-2xz}$$

MCQ (Multiple Choice Questions)

Q1. A current I is created by a narrow beam of protons moving in vacuum with constant velocity v. The magnitude of the Poynting vector $\vec{S}$ outside the beam at a radial distance r (much larger than the width of the beam) from the axis, are

(a) $|\vec{S}| = \frac{I}{4\pi^2\varepsilon_0 v r^2}$ (b) $|\vec{S}| = \frac{I}{4\pi^2\varepsilon_0 v r^4}$

(c) $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0 v r^2}$ (d) $|\vec{S}| = \frac{I^2}{4\pi^2\varepsilon_0 v r^4}$

Q2. Two electromagnetic waves are travelling along the z-direction in vacuum. The electric vectors of the two waves are

$$\vec{E}_1 = \hat{i}E_0\cos(kz-\omega t) + \hat{j}E_0\sin(kz-\omega t)$$

and $$\vec{E}_2 = \hat{i}E_0\sin(kz-\omega t) + \hat{j}E_0\cos(kz-\omega t)$$

The state of polarization of the wave resulting from the superposition of these two waves is:

(a) circular (b) plane (c) elliptical (d) unpolarised

Q3. The $\vec{E}$ field of a plane EM wave is $\vec{E}(z,t) = E_0\sin(kz+\omega t)\hat{y}$. The magnetic field of this wave is given by

(a) $\vec{B}(z,t) = \frac{kE_0}{\omega}\sin(kz+\omega t)\hat{x}$ (b) $\vec{B}(z,t) = -\frac{E_0}{k\omega}\sin(kz+\omega t)\hat{x}$

(c) $\vec{B}(z,t) = -\frac{E_0}{k\omega}\sin(kz+\omega t)\hat{z}$ (d) $\vec{B}(z,t) = \frac{kE_0}{\omega}\sin(kz+\omega t)\hat{z}$

Q4. The $\vec{B}$ field of a plane EM wave is $\vec{B}(y,t) = B_0\sin(ky-\omega t)\hat{z}$. The electric field of this wave is given by

(a) $\vec{E}(y,t) = \frac{B_0}{c}\sin(ky-\omega t)\hat{y}$ (b) $\vec{E}(y,t) = -cB_0\sin(ky-\omega t)\hat{y}$

(c) $\vec{E}(y,t) = \frac{B_0}{c}\sin(ky-\omega t)\hat{x}$ (d) $\vec{E}(y,t) = -cB_0\sin(ky-\omega t)\hat{x}$

Q5. The electric field $\vec{E}_1(r,t) = E_0(\hat{x} + i\hat{y})\exp\left[i(kz - \omega t)\right]$ and $\vec{E}_2(r,t) = E_0 e^{i(kz-\omega t)}\hat{x}$ of an electromagnetic wave propagating along positive z-direction, have state polarization.

(a) Plane polarized and circularly polarized

(b) Circularly polarized and plane polarized

(c) Elliptically polarized and plane polarized

(d) Both are plane polarized

Q6. The electric field of an electromagnetic wave is given by

$$\vec{E} = E_1 \cos(\omega t - kz)\hat{x} + E_2 \sin(\omega t - kz)\hat{y}$$

The state of polarization of the electromagnetic wave is:

(a) Circular, clockwise (b) circular, anti-clockwise

(d) Elliptical, clockwise (d) elliptical, anti-clockwise

Q7. Assume that a lamp radiates power P in free space uniformly in all directions. What is the magnitude of electric field strength at a distance r from the lamp?

(a) $\frac{P}{\pi c\varepsilon_0 r^2}$ (b) $\frac{P}{2\pi c\varepsilon_0 r^2}$ (c) $\sqrt{\frac{P}{2\pi c\varepsilon_0 r^2}}$ (d) $\sqrt{\frac{P}{\pi c\varepsilon_0 r^2}}$

Q8. A plane progressive wave is traveling with wave vector $\vec{k}$, $\hat{n}$ is the unit vector in the direction of amplitude. If the wave is longitudinal, then which of the following is correct?

(a) $\hat{n}\times\vec{k} \neq 0, \hat{n}\cdot\vec{k} = 0$ (b) $\hat{n}\times\vec{k} = 0, \hat{n}\cdot\vec{k} = 0$

(c) $\hat{n}\times\vec{k} = 0, \hat{n}\cdot\vec{k} \neq 0$ (d) $\hat{n}\times\vec{k} \neq 0, \hat{n}.\vec{k} \neq 0$

Q9. In a non-conducting medium characterized by $\varepsilon = \varepsilon_0, \mu = \mu_0$ and conductivity $\sigma = 0$, the electric field (in Vm^{-1}) is given by $\vec{E} = 20\sin\left[10^8 t - kz\right]\hat{y}$. The magnetic field $\vec{H}(in\ Am^{-1})$, is given by

(a) $20k\,\text{in}(10^8 t - kz)\hat{x}$ (b) $-\frac{20k}{10^8\mu_0}\sin(10^8 t - kz)\hat{y}$

(c) $-\frac{20k}{10^8\mu_0}\sin(10^8 t - kz)\hat{x}$ (d) $-20k\,\text{in}(10^8 t - kz)\hat{x}$

Q10. A plane electromagnetic wave has the magnetic field given by

$$\vec{B}(x,y,z,t) = B_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\hat{z},$$

where k is the wave number and $\hat{x}, \hat{y}$ and $\hat{z}$ are the cartesian unit vectors in x, y and z directions, respectively. Then the corresponding electric field is given by

(a) $\vec{E} = cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{x}+\hat{y})}{\sqrt{2}}$

(b) $\vec{E} = -cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{x}+\hat{y})}{\sqrt{2}}$

(c) $\vec{E} = -cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{x}-\hat{y})}{\sqrt{2}}$

(d) $\vec{E} = cB_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} + \omega t\right]\frac{(\hat{x}-\hat{y})}{\sqrt{2}}$

Q11. A plane electromagnetic wave has the magnetic field given by

$$\vec{B}(x,y,z,t) = B_0 \sin\left[(x+y)\frac{k}{\sqrt{2}} - \omega t\right]\hat{z},$$

where k is the wave number and $\hat{x}, \hat{y}$ and $\hat{z}$ are the cartesian unit vectors in x, y and z directions, respectively. Then the average Poynting vector is given by

(a) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{x}-\hat{y})}{\sqrt{2}}$ (b) $\frac{-cB_0^2}{2\mu_0}\frac{(\hat{x}-\hat{y})}{\sqrt{2}}$

(c) $\frac{cB_0^2}{2\mu_0}\frac{(\hat{x}+\hat{y})}{\sqrt{2}}$ (d) $\frac{-cB_0^2}{2\mu_0}\frac{(\hat{x}+\hat{y})}{\sqrt{2}}$

Q12. The electric field $\vec{\mathrm{E}}(\vec{r},t)$ for a circularly polarized electromagnetic wave propagating along the positive z direction is:

(a) $\vec{\mathrm{E}}(\vec{r},t) = \mathrm{E}_0(\hat{x}+\hat{y})\exp[i(kz-\omega t)]$ (b) $\vec{\mathrm{E}}(\vec{r},t) = \mathrm{E}_0(\hat{x}+i\hat{y})\exp[i(kz-\omega t)]$

(c) $\vec{\mathrm{E}}(\vec{r},t) = \mathrm{E}_0(\hat{x}+i\hat{y})\exp[i(kz+\omega t)]$ (d) $\vec{\mathrm{E}}(\vec{r},t) = \mathrm{E}_0(\hat{x}+\hat{y})\exp[i(kz+\omega t)]$

Q13. The state of polarization of light wave with the electric field vector

$$\vec{E} = \hat{x}E_0 \cos(kz - \omega t) + \hat{y}E_0 \cos(kz - \omega t)$$ is

(a) Circularly polarized

(b) Elliptically polarized with the major axis along x-axis

(c) Linearly polarized along z-direction

(d) Linearly polarized at 45^o to x-axis

Q14. A beam of light of frequency ω is reflected from a dielectric-metal interface at normal incidence. The refractive index of the dielectric medium is n and that of the metal is $n_2 = n(1 + i\rho)$. If the beam is polarised parallel to the interface, then the phase change experienced by the light upon reflection is

(a) $\tan(2/\rho)$ (b) $\tan^{-1}(1/\rho)$ (c) $\tan^{-1}(2/\rho)$ (d) $\tan^{-1}(2\rho)$

NAT (Numerical Answer Type)

Q15. Assume that a lamp radiates $5\,Watts$ of power in free space uniformly in all directions. Then the magnitude of electric field strength (in V/m) at a distance $1\,m$ from the lamp is…….

Q16. The plane electromagnetic wave in fee space in *SI* units are given by the expressions

$$E_x = 0,\ E_y = 2\cos\left[2\pi \times 10^{14}\left(t - \frac{x}{c}\right) + \frac{\pi}{2}\right] \text{ and } E_z = 0$$

The wavelength of the wave is……..$\times 10^{-6}\,m$

Q17. An electromagnetic wave with $\vec{E}(z,t) = E_0 \cos(\omega t - k z)\hat{x}$ is traveling in free space and crosses a disc of radius 2 m placed perpendicular to the z-axis. If $E_0 = 60\ V/m$, the average power, in Watt, crossing the disc along the z-direction is…….

Q18. The intensity of sunlight hitting the earth is about $1300\,W/m^2$. If sunlight strikes a perfect reflector, then the pressure it would exert will be…….$\times 10^{-6}\ N/m^2$

Q19. The intensity of sunlight hitting the earth is about $1300\,W/m^2$. If sunlight strikes a perfect absorber, then the pressure it would exert will be…….$\times 10^{-6}\ N/m^2$

Q20. Consider the superposition of two coherent electromagnetic waves whose electric field vectors are given by $\vec{E}_1 = \hat{i}E_0\cos\omega t$ and $\vec{E}_2 = \hat{j}E_0\cos\left(\omega t + \frac{\pi}{2}\right)$. The intensity of the resulting wave is given by $\frac{\varepsilon_0}{2}\left\langle E^2\right\rangle = \alpha\varepsilon_0 E_0^2$, where $\left\langle E^2\right\rangle$ is the time average of E^2. Then the value of α is…….

Q21. The refractive index of a medium in which the electric field of an electromagnetic wave is given in MKS units by $\vec{E} = E_0\cos\left(10^7 x + 10^7 y - 10^{15}t\right)\hat{z}$ is……..

Q22. An electromagnetically-shielded room is designed so that at a frequency $\omega = 10^7$ rad/s the intensity of the external radiation that penetrates the room is 1% of the incident radiation. If $\sigma = \frac{1}{2\pi}\times 10^6\,(\Omega m)^{-1}$ is the conductivity of the shielding material, its minimum thickness should be (given that $\ln 10 = 2.3$) ……..*mm*

Q23. An electromagnetic wave of frequency $10\,GHz$ is propagating through a conductor having conductivity $6\times 10^7\,(\Omega m)^{-1}$. Then skin depth of the conductor is…….. μm (Assume $\mu = \mu_0, \varepsilon = \varepsilon_0$)

MSQ (Multiple Select Questions)

Q24. Which of the following functions satisfy the wave equation (where symbols have their usual meaning)

(a) $f(z,t) = Ae^{-\alpha(z-vt)^2}$ (b) $f(z,t) = A\sin[\alpha(z-vt)]$

(c) $f(z,t) = Ae^{-\alpha(z^2+vt)}$ (d) $f(z,t) = \frac{A}{\alpha(z-vt)^2+1}$

Q25. Which of the following functions satisfy the wave equation (where symbols have their usual meaning)

(a) $f(z,t) = A\sin(bz)\cos(bvt)^3$ (b) $f(z,t) = A\sin[\alpha(z-vt)]$

(c) $f(z,t) = Ae^{-\alpha(z^2+vt)}$ (d) $f(z,t) = \frac{A}{\alpha(z-vt)^2+1}$

Q26. Consider the following statements:

(a) An electromagnetic wave incident upon a perfectly absorbing surface exerts pressure equal to that when the electromagnetic wave is incident upon a perfectly reflecting surface.

(b) The radiation pressure is directly proportional to the magnitude of Poynting vector.

(c) An electromagnetic wave incident upon a perfectly absorbing surface exerts pressure equal to half of that when the electromagnetic wave is incident upon a perfectly reflecting surface.

(d) The radiation pressure is not proportional to the magnitude of Poynting vector.

Which of the statements given above is/are correct?

Q27. A plane electromagnetic wave is propagating in a lossless dielectric. The electric field is given by

$$\vec{E}(x,y,z,t) = E_0\left(\hat{x} + \sqrt{2}A\hat{z}\right)\exp\left[ik_0\left\{-ct + \left(x + \sqrt{3}z\right)\right\}\right],$$

where c is the speed of light in vacuum, E_0, A and k_0 are constant and $\hat{x}$ and $\hat{z}$ are unit vectors along the x- and z-axes. n is the refractive index of the medium and A is some constant. Then which of the following are true?

(a) $n = 2$ (b) $n = 4$ (c) $A = \frac{1}{\sqrt{6}}$ (d) $A = -\frac{1}{\sqrt{6}}$

Q28. An electromagnetic wave propagates inside a conductor. Then which of the following statements are true.

(a) The amplitude of E and B decreases exponentially.

(b) There is phase delay between E and B

(c) E and B are perpendicular to each other

(d) Phase delay does not depend on particular conductor used

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: Let charge per unit length be λ, hence $I = \lambda u$ in z-direction.

The magnetic field at a distance r is $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$.

The electric field at a distance r is $\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r} = \frac{I}{2\pi\varepsilon_0 ur}\hat{r}$.

Hence Poynting vector $\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0} = \frac{I^2}{4\pi^2\varepsilon_0 ur^2}\hat{z}$

Ans. 2: (b)

Ans. 3: (a)

Solution: $\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} = -\frac{-k\hat{z}\times E_0\sin(kz+\omega t)\,\hat{y}}{\omega} = \frac{kE_0}{\omega}\sin(kz+\omega t)\,\hat{x} = \frac{kE_0}{\omega}\sin(kz+\omega t)\,\hat{x}$

Ans. 4: (d)

Solution: $\vec{E} = -\frac{c}{k}\left(\vec{k}\times\vec{B}\right) = -\frac{c}{k}\left[k\hat{y}\times B_0\sin\left(ky-\omega t\right)\hat{z}\right]$

$= -cB_0\sin\left(ky-\omega t\right)\hat{x} = -E_0\sin\left(ky-\omega t\right)\hat{x}$

Ans. 5: (b)

Solution: $\vec{E}_1(r,t) = E_0(\hat{x}+i\hat{y})[\cos(kz-\omega t)+i\sin(kz-\omega t)]$

Real part $= E_0\left[\cos(kz-\omega t)\,\hat{x} - \sin(kz-\omega t)\,\hat{y}\right]$

$E_x = E_0\cos(kz-\omega t)$, $E_y = -E_0\sin(kz-\omega t) \Rightarrow E_x^2 + E_y^2 = E_0^2$ (Circular polarization)

Ans. 6: (d)

Solution: $E_x = E_1\cos\left(\omega t - kz\right)$, $E_y = E_2\sin\left(\omega t - kz\right)$.

Thus resultant is elliptically polarized wave.

At $z = 0$, $E_x = E_1\cos\left(\omega t\right)$, $E_y = E_2\sin\left(\omega t\right)$

When $\omega t = 0$, $E_x = E_1$, $E_y = 0$ and when $\omega t = \frac{\pi}{2}$, $E_x = 0$, $E_y = E_2$

Ans. 7: (c)

Solution: $I=\frac{P}{A}=\frac{1}{2}\varepsilon_0 E_0^2 c=\frac{P}{4\pi r^2} \Rightarrow E_0^2=\frac{P}{2\pi\varepsilon_0 c r^2}\Rightarrow E_0=\sqrt{\frac{P}{2\pi\varepsilon_0 c r^2}}$

Ans. 8: (c)

Ans. 9: (c)

Solution: $\vec{H}=\frac{1}{\mu_0\omega}\vec{k}\times\vec{E}=\frac{1}{\mu_0 10^8}k\hat{z}\times 20\sin(10^8 t-kz)\hat{y}=-\frac{20k}{10^8\mu_0}\sin(10^8 t-kz)\hat{x}$

Ans. 10: (d)

Solution: $\vec{k}=\frac{k}{\sqrt{2}}(\hat{x}+\hat{y})$ and wave is propagating in $-\vec{k}$ direction.

$$\vec{E}=-\frac{c}{k}\left(\vec{k}\times\vec{B}\right)=-\frac{c}{k}\left[-\frac{k(\hat{x}+\hat{y})}{\sqrt{2}}\times B_0\sin\left\{\frac{(x+y)k}{\sqrt{2}}+\omega t\right\}\hat{z}\right]$$

$$\vec{E}=cB_0\sin\left[(x+y)\frac{k}{\sqrt{2}}+\omega t\right]\frac{(\hat{x}-\hat{y})}{\sqrt{2}}$$

Ans. 11: (c)

Solution: $\vec{k}=\frac{k}{\sqrt{2}}(\hat{x}+\hat{y})$ and $\vec{S}=\frac{cB_0^2}{2\mu_0}\hat{k}=\frac{cB_0^2}{2\mu_0}\times\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)=\frac{cB_0^2}{2\mu_0}\times\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$

Ans. 12: (b)

Solution: $\vec{\mathrm{E}}(\vec{r},t)=\mathrm{E}_0(\hat{x}+i\hat{y})\exp[i(kz-\omega t)]\Rightarrow E_x=\mathrm{E}_0\cos(kz-\omega t), E_y=-\mathrm{E}_0\sin(kz-\omega t)$

$\Rightarrow E_x^2+E_y^2=E_0^2(circular\ \ polarisation)$

Ans. 13: (d)

Ans. 14: (c)

Solution: Since $\tilde{E}_{0R}=\left(\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$ where $\tilde{\beta}=\frac{v_1}{v_2}=\frac{c/n}{c/n(1+i\rho)}=1+i\rho$

$$\Rightarrow \tilde{E}_{0R}=\left(\frac{-i\rho}{2+i\rho}\right)\tilde{E}_{0I}=\left(\frac{\rho e^{-i\pi/2}}{\sqrt{4+\rho^2}e^{i\theta}}\right)\tilde{E}_{0I}=\left(\frac{\rho}{\sqrt{4+\rho^2}}\right)e^{-i(\pi/2+\theta)}\tilde{E}_{0I}\ \text{ where }\ \tan\theta=\frac{\rho}{2}.$$

Thus phase change $\phi=-(\pi/2+\theta)\Rightarrow\tan\phi=\cot\theta=\frac{2}{\rho}\Rightarrow\phi=\tan^{-1}\left(\frac{2}{\rho}\right)$

NAT (Numerical Answer Type)

Ans. 15: **17**

Solution: $I=\frac{P}{A}=\frac{1}{2}\varepsilon_0 E_0^2 c=\frac{P}{4\pi r^2} \Rightarrow E_0^2=\frac{P}{2\pi\varepsilon_0 c r^2}\Rightarrow E_0=\sqrt{\frac{P}{2\pi\varepsilon_0 c r^2}}=\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{2P}{cr^2}}$

$$\Rightarrow E_0=\sqrt{\frac{1}{4\pi\varepsilon_0}\frac{2P}{cr^2}}=\sqrt{9\times10^9\frac{2\times5}{3\times10^8\times(1)^2}}=10\sqrt{3}\approx17$$

Ans. 16: **3**

Solution: $k=\frac{2\pi\times10^{14}}{c}=\frac{2\pi}{\lambda}\Rightarrow\lambda=\frac{c}{10^{14}}=\frac{3\times10^8}{10^{14}}=3\times10^{-6}m$

Ans. 17: **60**

$$I=\frac{P}{A}\Rightarrow P=IA=\frac{1}{2}\varepsilon_0 cE_0^2\times\pi r^2=60\ Watt\,.$$

Ans. 18: **8.66**

Solution: $P=\frac{2\mathrm{I}}{c}=\frac{2\times1300}{3\times10^8}=8.66\times10^{-6}N/m^2$

Ans. 19: **4.33**

Solution: $P=\frac{\mathrm{I}}{c}=\frac{1300}{3\times10^8}=4.33\times10^{-6}N/m^2$

Ans. 20: **0.5**

Solution: $E^2=E_1^2+E_2^2+2E_1E_2=E_0^2\cos^2\omega t+E_0^2\cos^2(\omega t+\phi)+2E_0^2\cos\omega t.\cos(\omega t+\phi)$

$$\Rightarrow\frac{\varepsilon_0}{2}\langle E^2\rangle=\frac{\varepsilon_0}{2}\left(\frac{E_0^2}{2}+\frac{E_0^2}{2}+2E_0^2.\frac{1}{2}\cos\phi\right)=\frac{\varepsilon_0}{2}E_0^2(1+\cos\phi)=\varepsilon_0E_0^2\cos^2(\phi/2)$$

$$\Rightarrow\frac{\varepsilon_0}{2}\langle E^2\rangle=\varepsilon_0E_0^2\cos^2(\pi/4)=\frac{1}{2}\varepsilon_0E_0^2$$

Ans. 21: **4.2**

Solution: $\omega=10^{15},\ k=\sqrt{2}\times10^7\Rightarrow v=\frac{\omega}{k}=\frac{10^8}{\sqrt{2}}\qquad\Rightarrow n=\frac{c}{v}=3.0\times\sqrt{2}=4.2$

Ans. 22: **2.3**

Solution: $I = I_0 e^{-2\kappa z} \Rightarrow z = \frac{1}{2\kappa}\ln\left(\frac{I_0}{I}\right)$

where $\frac{I_0}{I} = 100,\ \kappa = \sqrt{\frac{\sigma\mu\omega}{2}} = \sqrt{\frac{1}{2}\times\frac{1}{2\pi}\times 10^6 \times 4\pi \times 10^{-7} \times 10^7} = 10^3$

$\Rightarrow z = \frac{1}{2\times 10^3}\ln(100) = 2.3\, mm$

Ans. 23: **0.6**

Solution: $d = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{2}{2\times 3.14\times 10^{10}\times 6\times 10^7 \times 4\pi \times 10^{-7}}} = 6\times 10^{-7} m = 0.6\ \mu m$

MSQ (Multiple Select Questions)

Ans. 24: (a), (b), (d)

Ans. 25: (a), (b) and (d)

Ans. 26: (b) and (c)

Ans. 27: (a) and (d)

Solution: $\vec{E}(x,y,z,t) = E_0\left(\hat{x} + \sqrt{2}A\hat{z}\right)\exp\left[ik_0\left\{-ct + \left(x + \sqrt{3}z\right)\right\}\right]$.

Comparing with term $e^{i(\vec{k}\cdot\vec{r}-\omega t)} \Rightarrow \vec{k} = k_0\left(\hat{x} + \sqrt{3}\hat{z}\right)$ and $\omega = k_0 c$.

Since $v = \frac{\omega}{k} = \frac{k_0 c}{\sqrt{k_0^2 + 3k_0^2}} = \frac{c}{2} \Rightarrow$ Refractive index $n = 2$.

Since $\vec{k}\cdot\hat{n} = 0 \Rightarrow k_0\left(\hat{x} + \sqrt{3}\hat{z}\right)\cdot\left(\hat{x} + \sqrt{2}A\hat{z}\right) = 0 \Rightarrow k_0\left(1 + A\sqrt{6}\right) = 0 \Rightarrow A = -\frac{1}{\sqrt{6}}$

Ans. 28: (a), (b), (c)

Chapter -7

Applications of Electromagnetic waves

7.1 Reflection and Refraction at Dielectric Interface

7.1.1 Normal Incidence

Suppose *xy* plane forms the boundary between two linear media. A plane wave of frequency ω, traveling in the z-direction and polarized in the *x* direction, approaches the interface from the left then

Incident Wave

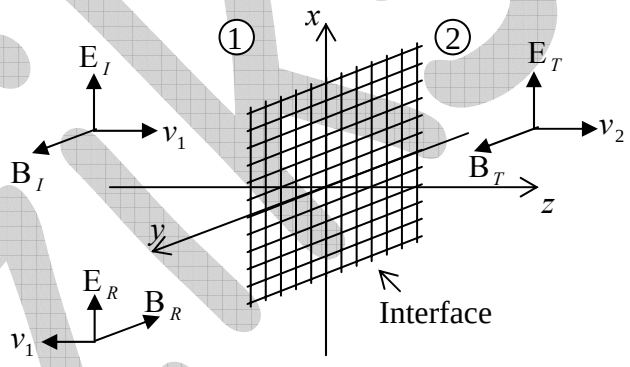
$$\left.\begin{aligned}\vec{\tilde{E}}_I(z,t) &= \tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_I(z,t) &= \frac{\tilde{E}_{0I}}{v_1}e^{i(k_1 z-\omega t)}\hat{y}\end{aligned}\right\}$$

Reflected Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_R(z,t) &= \tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_R(z,t) &= -\frac{\tilde{E}_{0R}}{v_1}e^{i(-k_1 z-wt)}\hat{y}\end{aligned}\right\}$$

Transmitted Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_T(z,t) &= \tilde{E}_{0T}e^{i(k_2 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_T(z,t) &= \frac{\tilde{E}_{0T}}{v_2}e^{i(k_2 z-\omega t)}\hat{y}\end{aligned}\right\}$$



At $z=0$, the combined field on the left $\tilde{E}_I+\tilde{E}_R$ *and* $\tilde{B}_I+\tilde{B}_R$, must join the fields on the right $\tilde{E}_T$ & $\tilde{B}_T$, in accordance with the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp}=\varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp}=B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel}=\vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel}=\frac{1}{\mu_2}\vec{B}_2^{\parallel}$

In this case there are no electric component perpendicular to the surface, so (i) & (ii) are trivial. However (iii) gives $\tilde{E}_{0I}+\tilde{E}_{0R}=\tilde{E}_{0T}$

While (iv) gives, $\frac{\tilde{E}_{0I}}{\mu_1 v_1}+\frac{\left(-\tilde{E}_{0R}\right)}{\mu_1 v_1}=\frac{\tilde{E}_{0T}}{\mu_2 v_2}$ or $\tilde{E}_{0I}-\tilde{E}_{0R}=\beta\tilde{E}_{0T}$

where $\beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}$.

Solving above two equations we get $\tilde{E}_{0R} = \left(\frac{1-\beta}{1+\beta}\right)\tilde{E}_{0I}$, $\tilde{E}_{0T} = \left(\frac{2}{1+\beta}\right)\tilde{E}_{0I}$.

If $\mu_1 = \mu_2 = \mu_0 \Rightarrow \quad \beta = \frac{v_1}{v_2} = \frac{n_2}{n_1}$ (For non-magnetic medium)

$$\Rightarrow \tilde{E}_{0R} = \left(\frac{v_2 - v_1}{v_2 + v_1}\right)\tilde{E}_{0I} \quad , \quad \tilde{E}_{0T} = \left(\frac{2v_2}{v_1 + v_2}\right)\tilde{E}_{0I}$$

Note: Reflected wave is *in phase* if $v_2 > v_1$ *or* $n_2 < n_1$ and *out of phase* if $v_2 < v_1$ *or* $n_2 > n_1$ **.**

In terms of indices of refraction the real amplitudes are

$$E_{0R} = \left|\frac{n_1 - n_2}{n_1 + n_2}\right| E_{0I} \quad , \quad E_{0T} = \left|\frac{2n_1}{n_1 + n_2}\right| E_{0I}.$$

Since Intensity $I = \frac{1}{2}\varepsilon v E_0^2$, then the ratio of the reflected intensity to the incident intensity is the *Reflection coefficient* $R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2$.

The ratio of the transmitted intensity to the incident intensity is the *Transmission coefficient*

$$T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \frac{4n_1 n_2}{(n_1 + n_2)^2} \quad \Rightarrow R + T = 1$$

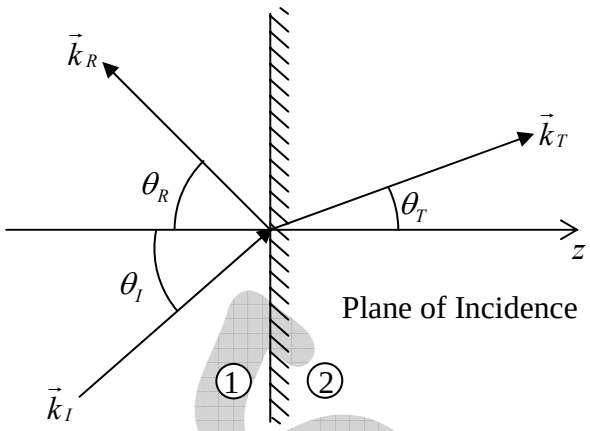
Example: Calculate the reflection coefficient for light at an air-to-dielectric interface $(\mu_1 = \mu_2 = \mu_0,\ n_1 = 1,\ n_2 = 1.5)$ at optical frequency $\omega = 4 \times 10^{15} s^{-1}$.

Solution: Reflection coefficient $R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1-1.5}{1+1.5}\right)^2 = 0.04$ *or* 4%

Thus only 4% of light is reflected and 96% is transmitted.

7.1.2 Oblique Incidence

In oblique incidence an incoming wave meets the boundary at an arbitrary angle θ_I. Of course, normal incidence is really just a special case of oblique incidence with $\theta_I = 0$. Suppose that a monochromatic plane wave of frequency ω, approaches the interface from the left then



Incident Wave

$$\vec{\tilde{E}}_I(\vec{r},t) = \vec{\tilde{E}}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_I(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_I \times \vec{\tilde{E}}_I\right)$$

Reflected Wave

$$\vec{\tilde{E}}_R(\vec{r},t) = \vec{\tilde{E}}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_R(\vec{r},t) = \frac{1}{v_1}\left(\hat{k}_R \times \vec{\tilde{E}}_R\right)$$

Transmitted Wave

$$\vec{\tilde{E}}_T(\vec{r},t) = \vec{\tilde{E}}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)}, \quad \vec{\tilde{B}}_T(\vec{r},t) = \frac{1}{v_2}\left(\hat{k}_T \times \vec{\tilde{E}}_T\right)$$

All three waves have the same frequency ω. The three wave numbers are related by ($\omega = kv$) as

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega \text{ or } k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T$$

The combined field in medium (1), $\vec{\tilde{E}}_I + \vec{\tilde{E}}_R$ *and* $\vec{\tilde{B}}_I + \vec{\tilde{B}}_R$, must join the fields $\vec{\tilde{E}}_T$ & $\vec{\tilde{B}}_T$ in medium (2), using the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp} = B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

First Law (Plane of Incidence)

The incident, reflected and transmitted wave vectors form a plane (called the plane of incidence), which also includes normal to the surface.

Second law (Law of Reflection)

The angle of incidence is equal to the angle of reflection i.e.

$$\theta_I = \theta_R$$

Third Law: (Law of Refraction, or Snell's law)

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

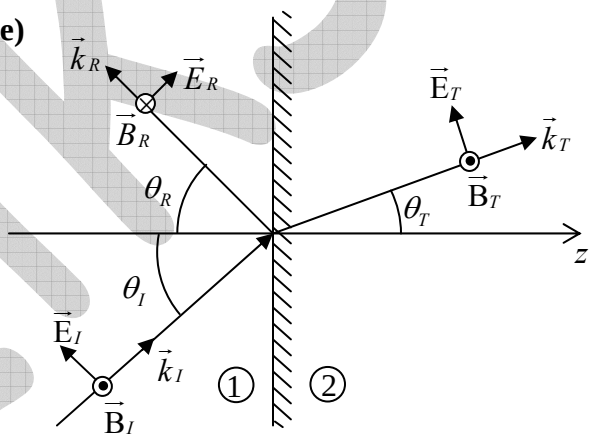
7.1.3 Fresnel's Relation (Parallel and Perpendicular Polarization)

Case-I: (Polarization in the Plane of Incidence)

Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$$\tilde{E}_{0R} = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)\tilde{E}_{0I} \quad \text{and} \quad \tilde{E}_{0T} = \left(\frac{2}{\alpha+\beta}\right)\tilde{E}_{0I}$$



where $\alpha = \dfrac{\cos\theta_T}{\cos\theta_I}$ *and* $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$

These are known as **Fresnel's equations**.

Notice that transmitted wave is always in phase with the incident one; the reflected wave is either in phase , if $\alpha > \beta$, or 180^0 out phase if $\alpha < \beta$.

The amplitudes of the transmitted and reflected waves depend on the angle of incidence, because α is a function of θ_I:

$$\alpha = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-\left(n_1/n_2\right)^2\sin^2\theta_I}}{\cos\theta_I}$$

Brewster's Angle

At *Brewster's angle* (θ_B) reflected light is completely extinguished when $\alpha = \beta$, or

$$\sin^2\theta_B = \frac{1-\beta^2}{\left(n_1/n_2\right)^2 - \beta^2}$$

For non-magnetic medium ($\mu_1 \cong \mu_2$), so $\beta \cong \frac{n_2}{n_1}$, $\sin^2\theta_B \cong \frac{\beta^2}{1+\beta^2}$, and hence

$$\tan\theta_B \approx \frac{n_2}{n_1} \quad and \quad \theta_T + \theta_B = 90^o$$

Thus at Brewster angle $\left(\theta_I = \theta_B\right)$ reflected and transmitted rays are perpendicular to each other.

Critical Angle

When light enters from denser to rarer medium ($n_1 > n_2$) then after a ***critical angle*** (θ_C) there is total internal reflection.

$$\frac{\sin 90^0}{\sin\theta_c} = \frac{n_1}{n_2} \quad \Rightarrow \quad \sin\theta_c = \frac{n_2}{n_1} \quad at \quad \theta_C, \quad \theta_T = 90^o$$

Reflection and Transmission Coefficient

The power per unit are striking the interface is $\vec{S}\hat{z}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I,$$

while reflected and transmitted intensities are

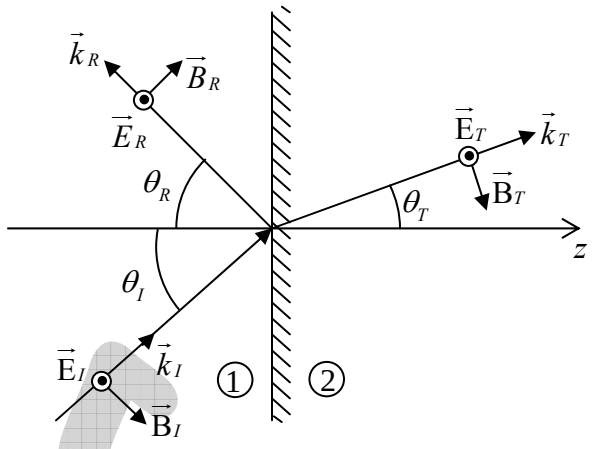
$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}^2 \cos\theta_R \text{ and } I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

Reflection coefficient $R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2$

Transmission coefficient $T = \frac{I_T}{I_I} = \frac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$

$$\Rightarrow R + T = 1$$

Case-II: (Polarization Perpendicular to plane of Incidence



Applying Boundary conditions, we get

Reflected and transmitted amplitudes

$$\hat{E}_{0R} = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\hat{E}_{0I} \quad and \quad \hat{E}_{0T} = \left(\frac{2}{1+\alpha\beta}\right)\hat{E}_{0I}$$

where $\alpha = \dfrac{\cos\theta_T}{\cos\theta_I}$ *and* $\beta = \dfrac{\mu_1 v_1}{\mu_2 v_2} = \dfrac{\mu_1 n_2}{\mu_2 n_1}$

In this case *Brewster's angle* (θ_B) is not possible i.e reflected light is never completely extinguished (since $\alpha\beta = 1$ is not possible).

Reflection and Transmission coefficient

The power per unit are striking the interface is $\vec{S}.\hat{z}$. Thus the incident intensity is

$$I_I = \frac{1}{2}\varepsilon_1 v_1 E_{0I}^2 \cos\theta_I ,$$

while reflected and transmitted intensities are

$$I_R = \frac{1}{2}\varepsilon_1 v_1 E_{0R}^2 \cos\theta_R \text{ and } I_T = \frac{1}{2}\varepsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

Reflection coefficient $R = \dfrac{I_R}{I_I} = \left(\dfrac{E_{0R}}{E_{0I}}\right)^2 = \left(\dfrac{1-\alpha\beta}{1+\alpha\beta}\right)^2$

Transmission coefficient $T = \dfrac{I_T}{I_I} = \dfrac{\varepsilon_2 v_2}{\varepsilon_1 v_1}\left(\dfrac{E_{0T}}{E_{0I}}\right)^2 \dfrac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\dfrac{2}{1+\alpha\beta}\right)^2$

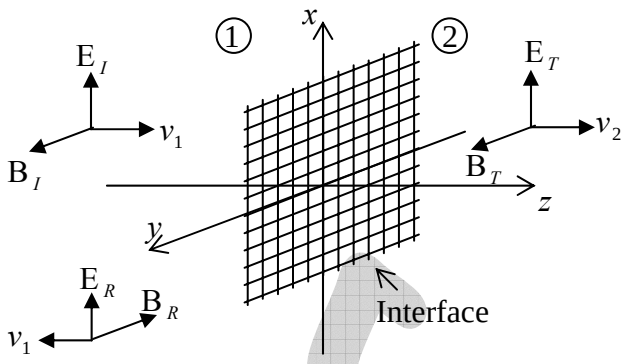
$$\Rightarrow R + T = 1$$

7.2 Reflection at Conducting Surface (Normal Incidence)

Suppose *xy* plane forms the boundary between a non-conducting linear medium (1) and a conductor (2). A plane wave of frequency ω, traveling in the z-direction and polarized in the *x* direction, approaches the interface from the left then

Incident Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_I(z,t) &= \tilde{E}_{0I}e^{i(k_1 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_I(z,t) &= \frac{\tilde{E}_{0I}}{v_1}e^{i(k_1 z-\omega t)}\hat{y}\end{aligned}\right\}$$



Reflected Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_R(z,t) &= \tilde{E}_{0R}e^{i(-k_1 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_R(z,t) &= -\frac{\tilde{E}_{0R}}{v_1}e^{i(-k_1 z-wt)}\hat{y}\end{aligned}\right\}$$

Transmitted Wave

$$\left.\begin{aligned}\vec{\tilde{E}}_T(z,t) &= \tilde{E}_{0T}e^{i(\tilde{k}_2 z-\omega t)}\hat{x}\\ \vec{\tilde{B}}_T(z,t) &= \frac{\tilde{k}_2}{\omega}\tilde{E}_{0T}e^{i(\tilde{k}_2 z-\omega t)}\hat{y}\end{aligned}\right\}$$

where $\tilde{k}_2 = k_2 + i\kappa_2$ where k_2 and κ_2 are real and imaginary part of $\tilde{k}_2$.

$$k_2 = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}+1\right]^{1/2} \quad and \quad \kappa_2 = \omega\sqrt{\frac{\varepsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}-1\right]^{1/2}$$

At $z=0$, the combined field on the left $\tilde{E}_I + \tilde{E}_R$ *and* $\tilde{B}_I + \tilde{B}_R$, must join the fields on the right $\tilde{E}_T$ & $\tilde{B}_T$, in accordance with the ***boundary conditions***

(i) $\varepsilon_1 E_1^{\perp} = \varepsilon_2 E_2^{\perp}$ (ii) $B_1^{\perp} = B_2^{\perp}$ (iii) $\vec{E}_1^{\parallel} = \vec{E}_2^{\parallel}$ (iv) $\frac{1}{\mu_1}\vec{B}_1^{\parallel} = \frac{1}{\mu_2}\vec{B}_2^{\parallel}$

In this case there are no electric component perpendicular to the surface, so (i) & (ii) are trivial.

However (iii) gives $\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$

While (iv) gives, $\frac{\tilde{E}_{0I}}{\mu_1 v_1} + \frac{\left(-\tilde{E}_{0R}\right)}{\mu_1 v_1} = \frac{\tilde{k}_2}{\mu_2\omega}\tilde{E}_{0T}$ or $\tilde{E}_{0I} - \tilde{E}_{0R} = \tilde{\beta}\tilde{E}_{0T}$

where $\tilde{\beta} = \dfrac{\mu_1 v_1}{\mu_2 \omega}\tilde{k}_2$.

Solving above two equations we get $\tilde{E}_{0R} = \left(\dfrac{1-\tilde{\beta}}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$, $\tilde{E}_{0T} = \left(\dfrac{2}{1+\tilde{\beta}}\right)\tilde{E}_{0I}$.

Note: (i) For a perfect conductor $(\sigma = \infty)$, $k_2 = \infty \Rightarrow \tilde{\beta} = \infty$. Thus

$$\tilde{E}_{0R} = -\tilde{E}_{0I},\ \tilde{E}_{0T} = 0.$$

In this case wave is totally reflected, with a 180^0 phase shift.

(ii) For good conductor $(\sigma \gg \omega\varepsilon)$, $k_2 \cong \kappa_2 = \sqrt{\dfrac{\sigma\omega\mu_2}{2}}$.

$$\tilde{\beta} = \frac{\mu_1 v_1}{\mu_2 \omega}\sqrt{\frac{\sigma\omega\mu_2}{2}}(1+i) = \mu_1 v_1\sqrt{\frac{\sigma}{2\omega\mu_2}}(1+i) \Rightarrow \tilde{\beta} = \gamma(1+i) \text{ where } \gamma = \mu_1 v_1\sqrt{\frac{\sigma}{2\omega\mu_2}}.$$

Reflection Coefficient

$$R = \frac{I_R}{I_I} = \left|\frac{\tilde{E}_{0R}}{\tilde{E}_{0I}}\right|^2 = \left|\frac{1-\tilde{\beta}}{1+\tilde{\beta}}\right|^2 = \left(\frac{1-\gamma-i\gamma}{1+\gamma+i\gamma}\right)\left(\frac{1-\gamma+i\gamma}{1+\gamma-i\gamma}\right) = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2}$$

Example: Calculate the reflection coefficient for light at an air-to-silver interface $\left(\mu_1 = \mu_2 = \mu_0,\ \varepsilon_1 = \varepsilon_0, \sigma = 6\times10^7\,\Omega^{-1}m^{-1}\right)$ at optical frequency $\omega = 4\times10^{15}\,s^{-1}$.

Solution: $\gamma = \mu_0 c\sqrt{\dfrac{\sigma}{2\omega\mu_0}} = c\sqrt{\dfrac{\sigma\mu_0}{2\omega}} = \left(3\times10^8\right)\sqrt{\dfrac{\left(6\times10^7\right)\left(4\pi\times10^{-7}\right)}{2\left(4\times10^{15}\right)}} = 29$

Reflection coefficient

$$R = \frac{(1-\gamma)^2+\gamma^2}{(1+\gamma)^2+\gamma^2} = \frac{(28)^2+29^2}{(30)^2+29^2} = 0.93 \quad or \quad 93\%.$$

Thus 93% of light is reflected and only 7% is transmitted.

MCQ (Multiple Choice Questions)

Q.1 Unpolarized light falls from air to a planer air-glass interface (refractive index of glass is1.5) and the reflected light is observed to be plane polarized. The polarized vector and the angle of incidence θ_1 are

(a) Perpendicular to the plane of incidence and $\theta_1 = 42^0$

(b) Parallel to the plane of incidence and $\theta_1 = 56^0$

(c) Perpendicular to the plane of incidence and $\theta_1 = 56^0$

(d) Parallel to the plane of incidence and $\theta_1 = 52^0$

Q2. For normal incident at an air-glass interface with $n = 1.5$ the fraction of energy reflected is given by

(a) 0.40 (b) 0.20 (c) 0.16 (d) 0.04

Q3. The Fresnel relations between the amplitudes of incident and reflected electromagnetic waves at an interface between air and a dielectric of refractive index μ, are

$$E_{\parallel}^{reflected} = \frac{\cos r - \mu \cos i}{\cos r + \mu \cos i} E_{\parallel}^{incident} \text{ and } E_{\perp}^{reflected} = \frac{\mu \cos r - \cos i}{\mu \cos r + \cos i} E_{\perp}^{incident}$$

The subscripts $\parallel$ and $\perp$ refer to polarization, parallel and normal to the plane of incidence respectively. Here i and r are the angles of incidence and refraction respectively. The condition for the reflected ray to be completely polarized is

(a) $\mu \cos i = \cos r$ (b) $\cos i = \mu \cos r$

(c) $\mu \cos i = -\cos r$ (d) $\cos i = -\mu \cos r$

Q4. A plane electromagnetic wave traveling in vacuum is incident normally on a non-magnetic, non-absorbing medium of refractive index (n). The incident (E_i), reflected (E_r) and transmitted (E_t) electric fields are given as,

$$E_i = E\,exp\left[i\left(kz-\omega t\right)\right],\ E_r = E_{0r}exp\left[i\left(k_r z-\omega t\right)\right],\ E_t = E_{0t}exp\left[i\left(k_t z-\omega t\right)\right]$$

If $E = 2\ V/m$ and $n = 1.5$ then the application of appropriate boundary conditions leads to

(a) $E_{0r} = -\frac{3}{5}V/m, E_{0t} = \frac{7}{5}V/m$

(b) $E_{0r} = -\frac{1}{5}V/m, E_{0t} = \frac{8}{5}V/m$

(c) $E_{0r} = -\frac{2}{5}V/m, E_{0t} = \frac{8}{5}V/m$

(d) $E_{0r} = \frac{4}{5}V/m, E_{0t} = \frac{6}{5}V/m$

Q5. A plane electromagnetic wave of frequency ω is incident normally on an air-dielectric interface. The dielectric is linear, isotropic, non-magnetic and its refractive index is n. The reflectance (R) and transmittance (T) from the interface are

(a) $R = \left(\frac{n-1}{n+1}\right)^2,\quad T = \frac{4n}{(n+1)^2}$

(b) $R = -\left(\frac{n-1}{n+1}\right),\quad T = \frac{2}{(n+1)^2}$

(c) $R = \left(\frac{n-1}{n+1}\right)^3,\quad T = \frac{4n^3}{(n+1)^3}$

(d) $R = \left(\frac{(n-1)^2}{n+1}\right),\quad T = \frac{4n^2}{(n+1)^2}$

Q6. A plane wave traveling with a velocity c_1 in a medium of density ρ_1 is incident normally on an interface with another medium of density ρ_2 in which the wave velocity is c_2. If the fraction of energy transmitted is $\frac{4\rho_1\rho_2 c_1 c_2}{\left(\rho_1 c_1+\rho_2 c_2\right)^2}$. The fraction of energy reflected is equal to (assuming no absorption):

(a) $\frac{\rho_1 c_1-\rho_2 c_2}{\rho_1 c_1+\rho_2 c_2}$

(b) $\frac{\rho_1 c_1-\rho_2 c_2}{\left(\rho_1 c_1+\rho_2 c_2\right)^2}$

(c) $\frac{\left(\rho_1 c_1-\rho_2 c_2\right)^2}{\rho_1 c_1+\rho_2 c_2}$

(d) $\left[\frac{\rho_1 c_1-\rho_2 c_2}{\rho_1 c_1+\rho_2 c_2}\right]^2$

Q7. For a particular medium θ_C is the critical angle for total internal reflection and θ_B is the Brewster angle for the same medium. Then which one of the following relationships will hold?

(a) $\sin\theta_C = \cot\theta_B$ (b) $\cos\theta_C = \tan\theta_B$

(c) $\tan\theta_C = \sin\theta_B$ (d) $\cot\theta_C = \cos\theta_B$

NAT (Numerical Answer Type)

Q8. A plane electromagnetic wave traveling in free space is incident normally on a glass plate of refractive index 3/2. If there is no absorption by the glass, its transmittivity is…….

Q9. Unpolarized light is incident on an air-dielectric interface. The interface is the xy plane and the plane of incidence is yz plane. The electric field of the reflected light is given by $\vec{E} = E_0\hat{x}\exp\left\{\frac{ik}{2}\left(\sqrt{3}y + z\right) - i\omega t\right\}$, where k is the propagation constant in air and ω is the angular frequency of the light. Assume magnetic permeability $\mu = \mu_0$. The dielectric constant of the second medium is…….

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Solution: $\theta_1 = \theta_B = \tan^{-1}(\mu) = 56^0$

Ans. 2: (d)

Solution: $R = \left(\frac{n_1 - n_2}{n_1 + n_2}\right)^2 = \left(\frac{1-1.5}{1+1.5}\right)^2 = \frac{1}{4} \times \frac{4}{25} = 0.04$

Ans. 3: (a)

Ans. 4: (c)

Solution: $E_{0R} = \left(\frac{n_1 - n_2}{n_1 + n_2}\right) E_{0I} \Rightarrow E_{0R} = \left(\frac{1-1.5}{1+1.5}\right) \times 2 = -\frac{1}{2.5} = -\frac{2}{5} = -0.40 \ V/m$

$$E_{0R} = \left(\frac{2n_1}{n_1 + n_2}\right) E_{0I} \Rightarrow E_{0R} = \left(\frac{2}{1+1.5}\right) \times 2 = \frac{4}{2.5} = \frac{8}{5} = 1.60 \ V/m$$

Ans. 5: (a)

Ans. 6: (d)

Solution: $R + T = 1 \Rightarrow R = 1 - T = 1 - \frac{4\rho_1\rho_2 c_1 c_2}{(\rho_1 c_1 + \rho_2 c_2)^2} = \left[\frac{\rho_1 c_1 - \rho_2 c_2}{\rho_1 c_1 + \rho_2 c_2}\right]^2$

Ans. 7: (a)

Solution: Let us assume light enters from rarer medium (2) to denser medium (1).

Then $\tan\theta_B = \frac{n_1}{n_2}$

At same interface $\Rightarrow \sin\theta_C = \frac{n_2}{n_1}$.

$$\Rightarrow \tan\theta_B = \frac{1}{\sin\theta_c} \Rightarrow \sin\theta_C = \cot\theta_B$$

NAT (Numerical Answer Type)

Ans. 8: **96%**

Solution: $R=\left(\frac{n_1-n_2}{n_1+n_2}\right)^2=\left(\frac{1-3/2}{1+3/2}\right)^2=\frac{1}{4}\times\frac{4}{25}=.04\ or\ 4\%\Rightarrow T=1-R=0.96\ or\ 96\%$

Ans. 9: **3**

Solution: Since reflected light is plane polarized, then angle of reflection is Brewster angle $\tan\theta_R=\tan\theta_B=\left(\frac{n_2}{n_1}\right)=n_2$.

$$\vec{k}=\frac{k}{2}\left(\sqrt{3}\hat{y}+\hat{z}\right)\Rightarrow\tan\theta_B=\left(\frac{k_y}{k_z}\right)=\sqrt{3}\Rightarrow n_2=\sqrt{\varepsilon_r}=\sqrt{3}\Rightarrow\varepsilon_r=3$$

Chapter - 8

Potential and Field Formulation for Time Varying Fields

1.1 Scalar and Vector Potentials

We want general solution to Maxwell's Equations,

(i) $\vec{\nabla}\cdot\vec{\mathrm{E}}=\frac{\rho}{\varepsilon_0}$ (Gauss' Law),

(ii) $\vec{\nabla}\cdot\vec{B}=0$ (No name),

(iii) $\vec{\nabla}\times\vec{\mathrm{E}}=-\frac{\partial\vec{B}}{\partial t}$ (Farday's Law),

(iv) $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$ (Ampere's law).

Given $\rho\left(\vec{r},t\right)$ and $\vec{J}\left(\vec{r},t\right)$ what are the fields $\vec{E}\left(\vec{r},t\right)$ and $\vec{B}\left(\vec{r},t\right)$?

From equation (ii) we can still write $\boxed{\vec{B}=\vec{\nabla}\times\vec{A}}$

Putting this into Faraday's Law (iii)

$$\vec{\nabla}\times\vec{\mathrm{E}}=-\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{A}\right)\Rightarrow\vec{\nabla}\times\left(\vec{\mathrm{E}}+\frac{\partial\vec{A}}{\partial t}\right)=0\Rightarrow\vec{E}+\frac{\partial\vec{A}}{\partial t}=-\vec{\nabla}V$$

$$\Rightarrow\quad\boxed{\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}}$$

Now from equation (i)

$$\nabla^2V+\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{A}\right)=-\frac{\rho}{\varepsilon_0}\qquad\text{.....(1)}$$

From equation (iv)

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\mu_0\vec{J}-\mu_0\varepsilon_0\vec{\nabla}\left(\frac{\partial V}{\partial t}\right)-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2};\qquad\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}.\vec{A}\right)-\nabla^2\vec{A}$$

$$\Rightarrow\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}\qquad\text{.....(2)}$$

Equations (1) and (2) contain all the information of Maxwell's equations. Thus we need to calculate only four components (one for V and three for $\vec{A}$) instead of six components (three for $\vec{E}$ and three for $\vec{B}$).

Example: In an infinite straight wire constant current I_0 is turned on abruptly at $t=0$. Then corresponding retarded potentials are given by

$$V(\vec{r},t)=0, \qquad \vec{A}(\vec{r},t)=\frac{\mu_0 I_0}{2\pi}\ln\left(\frac{ct+\sqrt{(ct)^2-r^2}}{r}\right)\hat{z}$$

Find the fields corresponding to these potentials.

Solution: The electric field is $\vec{E}(r,t)=-\frac{\partial\vec{A}}{\partial t}=-\frac{\mu_0 I_0 c}{2\pi\sqrt{(ct)^2-r^2}}\hat{z}$

And the magnetic field is $\vec{B}(r,t)=\vec{\nabla}\times\vec{A}=-\frac{\partial A_z}{\partial r}\hat{\phi}=\frac{\mu_0 I_0}{2\pi r}\frac{ct}{\sqrt{(ct)^2-r^2}}\hat{\phi}$

1.2 Gauge Transformation

Suppose we have two sets of potentials, $(V,\vec{A})$ and $(V',\vec{A}')$, which correspond to the same electric and magnetic fields.

Thus $\vec{A}'=\vec{A}+\vec{\alpha}$ and $V'=V+\beta$.

Since $\vec{B}'=\vec{\nabla}\times\vec{A}'=\vec{B}\Rightarrow\vec{\nabla}\times\vec{\alpha}=0\Rightarrow\vec{\alpha}=\vec{\nabla}\lambda.$

Two potentials also give the same $\vec{E}$,

So $$\vec{E}'=-\vec{\nabla}V'-\frac{\partial\vec{A}'}{\partial t}=\vec{E}\Rightarrow\vec{\nabla}\beta+\frac{\partial\vec{\alpha}}{\partial t}=0,$$

or $\vec{\nabla}\left(\beta+\frac{\partial\lambda}{\partial t}\right)=0\Rightarrow\beta+\frac{\partial\lambda}{\partial t}=k(t)\Rightarrow\beta=-\frac{\partial\lambda}{\partial t}+k(t)$

Actually, we might as well absorb $k(t)$ into λ, defining a new λ by adding $\int_0^t k(t')dt'$ to the old one. This will not affect the gradient of λ; it just adds $k(t)$ to $\frac{\partial\lambda}{\partial t}$.

It follows that $\left.\begin{aligned}\vec{A}' &= \vec{A}+\vec{\nabla}\lambda \\ V' &= V-\frac{\partial\lambda}{\partial t}\end{aligned}\right\}$

Conclusion: For any old scalar function λ, we can add $\vec{\nabla}\lambda$ to $\vec{A}$, provided we simultaneously subtract $\frac{\partial\lambda}{\partial t}$ from *V*. None of these will affect the physical quantities $\vec{E}$ and $\vec{B}$. Such changes in V *and* $\vec{A}$ are called **gauge transformations.**

1.3 Coulomb and Lorentz Gauge

Coulomb Gauge reads $\vec{\nabla}\cdot\vec{A}=0$.

Lorentz Gauge condition is $\vec{\nabla}\cdot\vec{A}=-\mu_0\varepsilon_0\frac{\partial V}{\partial t}$.

Since $\left(\nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}\right)-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}+\mu_0\varepsilon_0\frac{\partial V}{\partial t}\right)=-\mu_0\vec{J}$ and $\nabla^2V+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)=-\frac{\rho}{\varepsilon_0}$,

Using Lorentz Gauge condition

$$\Rightarrow \nabla^2\vec{A}-\mu_0\varepsilon_0\frac{\partial^2\vec{A}}{\partial t^2}=-\mu_0\vec{J} \text{ and } \nabla^2V-\mu_0\varepsilon_0\frac{\partial^2V}{\partial t^2}=-\frac{\rho}{\varepsilon_0}.$$

The virtue of the Lorentz gauge is that it treats V *and* $\vec{A}$ on an equal footing: the same differential operator $\nabla^2-\mu_0\varepsilon_0\frac{\partial^2}{\partial t^2}\equiv\Box^2$ (called the **d' Alembertian**) occurs in both equations:

$$\text{(i)}\ \Box^2V=-\frac{\rho}{\varepsilon_0}, \qquad \text{(ii)}\ \Box^2\vec{A}=-\mu_o\vec{J}$$

Example: For a vector potential $\vec{A}$ the divergence of $\vec{A}$ is $\vec{\nabla}\cdot\vec{A}=\frac{\mu_0}{4\pi}\frac{Q_0}{r^2}$ where Q_0 is a constant of appropriate dimensions. Find the corresponding scalar potential $V\left(\vec{r},t\right)$ that makes $\vec{A}$ and V Lorentz gauge invariant.

Solution: $\vec{\nabla}\cdot\vec{A}=-\mu_0\varepsilon_0\frac{\partial V}{\partial t}\Rightarrow\frac{\mu_0}{4\pi}\frac{Q_0}{r^2}=-\mu_0\varepsilon_0\frac{\partial V}{\partial t}\Rightarrow V=-\frac{1}{4\pi\varepsilon_0}\frac{Q_0t}{r^2}$

Chapter - 9

Simple AC Circuits with R, L and C Components

9.1 Kirchhoff's Law

9.1.1 Kirchhoff's Voltage Law (KVL)

Kirchhoff's voltage law states that the voltage applied to a closed circuit equals the sum of the voltage drops in that circuit. In series circuits it can be expressed as follows:

Voltage applied = sum of voltage drops

$$V_A = V_1 + V_2 + V_3$$

where V_A is the applied voltage and V_1, V_2 and V_3 are voltage drops.

Another way of stating KVL is that the algebraic sum of the voltage rises and voltage drops must be equal to zero. A voltage source or emf is considered a voltage rise; a voltage across a resistor is a voltage drop.

Voltage applied – sum of voltage drops = 0

Substitute letters: $V_A - V_1 - V_2 - V_3 = 0$ or $V_A - (V_1 + V_2 + V_3) = 0$

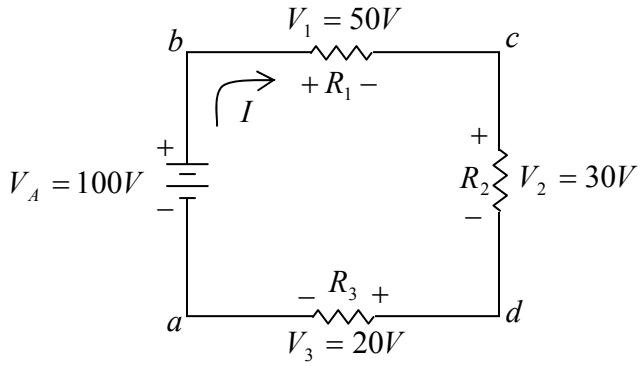
Using a symbol, Σ, we have

$$\Sigma V = V_A - V_1 - V_2 - V_3 = 0$$

in which ΣV, the algebraic sum of all the voltages around any closed circuit, equals zero.

We assign a + sign to a voltage rise and a – sign to a voltage drop for the $\Sigma V = 0$ formula (as shown in figure below). In tracing voltage drops around a circuit, start at the negative terminal of the voltage source. The path from the negative terminal to the positive terminal of the source is voltage rise. In figure if we start at point *a*, the negative terminal of the battery, and move around the circuit in the direction *abcda*, we go through V_A from – to + and V_A = +100V. If we start at point *b* and move in the opposite direction *badcb*, we go through V_A from + to – and V_A = –100 V. The voltage drop across any resistance will be negative (-) if we trace it in the + to – direction. Thus in figure, if we trace the circuit in the direction *abcda*, V_1 = –50V, V_2 = –30V and V_3 = –20V. The voltage drop will be positive (+) if we go through the resistance in the – to + direction. So in tracing the circuit in the direction *abcda*, we have

$$\Sigma V = V_A - V_1 - V_2 - V_3 = 0$$



$$\Sigma V = V_A - V_1 - V_2 - V_3$$
$$= 100 - 50 - 30 - 20$$
$$= 100 - 100$$
$$= 0$$

Figure: Illustration of ΣV = 0

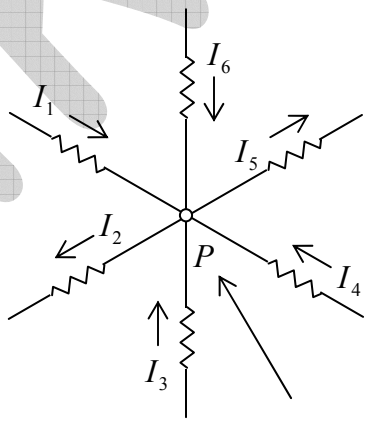
9.1.2 Kirchhoff's Current Law (KCL)

Kirchhof's current law states that the *sum of the currents entering a junction is equal to the sum of the currents leaving* the junction. Suppose we have six currents leaving and entering a common junction or point, shown as *P*. This common point is also called a node.

Substitute letters:

$$I_1 + I_3 + I_4 + I_6 = I_2 + I_5$$

If we consider that the currents flowing toward a junction are positive (+) and those currents flowing away from the same junction are negative (-), then this law also states that the algebraic sum of all the currents meeting at a common junction is zero. Using the symbol Σ, we have



Common point, junction, or node

Figure: Currents at a common point

$$\Sigma I = 0$$

where ΣI, the algebraic sum of all the currents at the common point is zero.

$$I_1 - I_2 + I_3 + I_4 - I_5 + I_6 = 0$$

If the negative terms are transposed to the right side of the equal sign, we would have the same form as the original equation.

9.1.3 Mesh Current

A simplification of Kirchhoff's laws is a method that makes use of mesh or loop currents. A mesh is any closed path of a circuit. It does not matter whether the path contains a voltage source. In solving a circuit with mesh currents, we first must decide which paths will be the meshes. Then we assign a mesh current to each mesh. For convenience, mesh currents are usually assigned in a clockwise direction. Kirchhoff's voltage law is then applied about the path of each mesh. The resulting equations determine the unknown mesh currents. From these currents, the current or voltage of any resistor can be found.

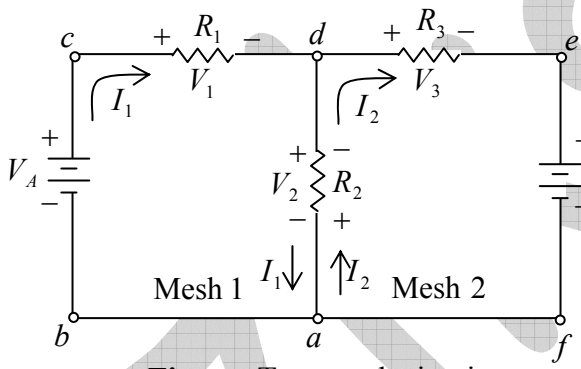


Figure: Two-mesh circuit

In figure shown above, we have a two-mesh circuit marked mesh 1 and mesh 2. Mesh 1 is path *abcda* and mesh 2 is path *adefa*. All voltage sources and resistances are known. A procedure for finding mesh currents I_1 and I_2 is as follows:

Step 1: After the meshes are selected, show the direction of mesh currents I_1 and I_2 in a clockwise direction. Mark the voltage polarity across each resistor, consistent with the assumed current. Remember that conventional current flow in a resistor produces positive polarity where the current enters.

Step 2: Apply Kirchhoff's voltage law, $\Sigma V = 0$, around each mesh. Trace each mesh in the direction of mesh current. Note that there are two different currents (I_1, I_2) flowing in opposite directions through the same resistor, R_2, which is common to both meshes. For this reason two sets of polarities are shown by R_2 (as shown in figure). Trace mesh 1 in direction *abcda*.

$$+V_A - I_1R_1 - I_1R_2 + I_2R_2 = 0$$

$$+V_A - I_1 (R_1 + R_2) + I_2R_2 = 0$$

or $$+ I_1 (R_1 + R_2) - I_2R_2 = V_A \qquad \text{.....(1)}$$

Note that in the first expression I_2R_2 is + since we go through a voltage drop from – to +.

Trace mesh 2 in direction *adefa*.

$$- I_2R_2 + I_1R_2 - I_2R_3 - V_B = 0$$

$$+ I_1R_2 - I_2 (R_2 + R_3) = V_B \qquad \text{.....(2)}$$

Note that I_1R_2 is a + voltage drop since we go through a voltage drop from – to +.

Step 3: Find I_1 and I_2 by solving equations (1) and (2) simultaneously.

Step 4: When mesh currents are known, find all resistor voltage drops by using Ohm's law.

Step 5: Check the solution of mesh currents by tracing mesh *abcdefa*.

$$V_A - I_1R_1 - I_2R_3 - V_B = 0$$

9.2 Alternating Current

9.2.1 Sine Wave

The instantaneous value of voltage at any point on the sine wave is expressed by the equation $v = V_M \sin\theta$

where v = instantaneous value of voltage, V , V_M = maximum value of voltage, V

θ = angle

9.2.2 Alternating Current

When a sine wave of alternating voltage is connected across a load resistance, the current that flows in the circuit is also a sine wave as shown in figure below.

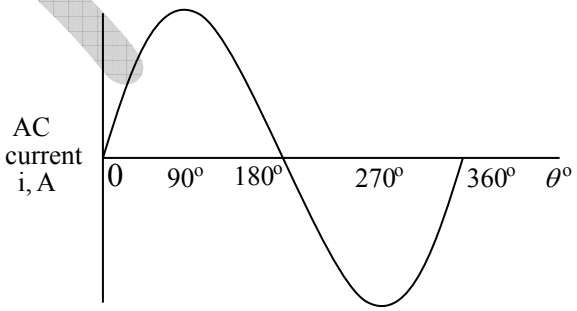


Figure: One cycle of alternating current

9.2.3 Frequency and Period

The number of cycles per second is called frequency. It is indicated by the symbol f and is expressed in hertz (Hz). One cycle per second equals one hertz. Thus 60 cycles per second (sometimes abbreviated *cps*) equals 60 Hz. A frequency of 2 Hz (figure *b*) is twice the frequency of 1 Hz (figure *a*).

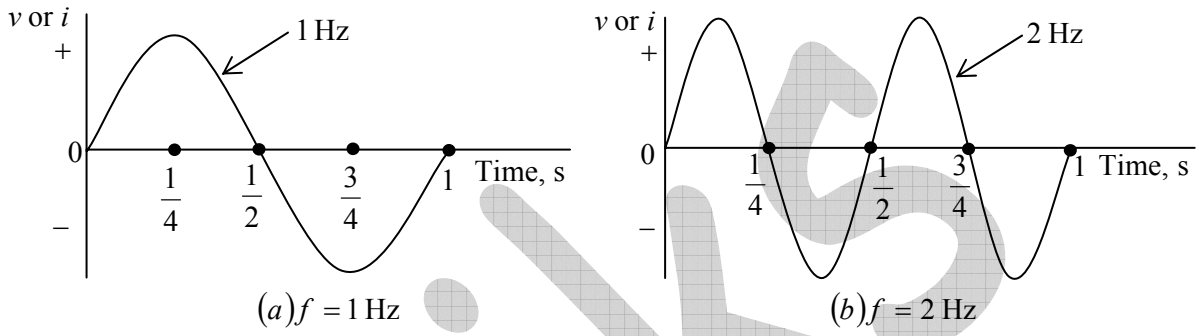


Figure: Comparison of frequencies

The amount of time for the completion of 1 cycle is the period. It is indicated by the symbol T for time and is expressed in seconds (s). Frequency and period are reciprocals of each other.

$$f = \frac{1}{T} \quad \text{or} \quad T = \frac{1}{f}$$

The higher the frequency, the shorter the period.

The angle of 360° represents the time for 1 cycle, or the period T. So we can show the horizontal axis of the sine wave in units of either electrical degrees or seconds.

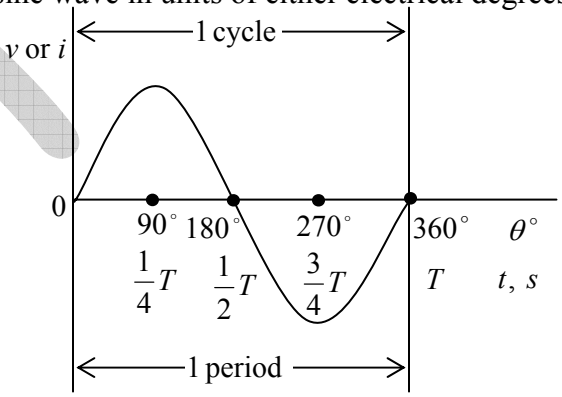


Figure: Relationship between electrical degrees and time

The wavelength λ is the length of one complete wave or cycle. It depends upon the frequency of the periodic variation and its velocity of transmission. Expressed as a formula,

$$\lambda = \frac{\text{velocity}}{\text{frequency}}$$

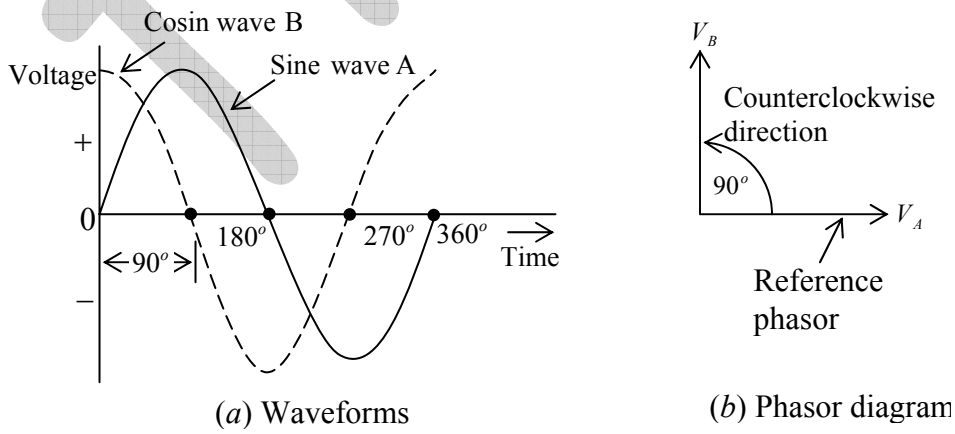
For electromagnetic radio waves, the velocity in air or a vacuum is 3×10^8 m/s, which is the speed of light. $\lambda = \frac{c}{f}$

where λ = wavelength, m , c = speed of light 3×10^8 m/s, a constant

f = radio frequency, Hz

9.2.4 Phase Relationships

The phase angle between two waveforms of the same frequency is the angular difference at a given instant of time. As an example, the phase angle between waves B and A (figure *a*) is 90°. Take the instant of time at 90°. The horizontal axis is shown in angular units of time. Wave B starts at maximum value and reduces to zero value at 90°, while wave A starts at zero and increases to maximum value at 90°. This 90° phase angle between waves B and A is maintained throughout the complete cycle and all successive cycles. At any instant of time, wave B has the value that wave A will have 90° later. Wave B is a cosine wave because it is displaced 90° from wave A, which is a sine wave. Both waveforms are called sinusoids.

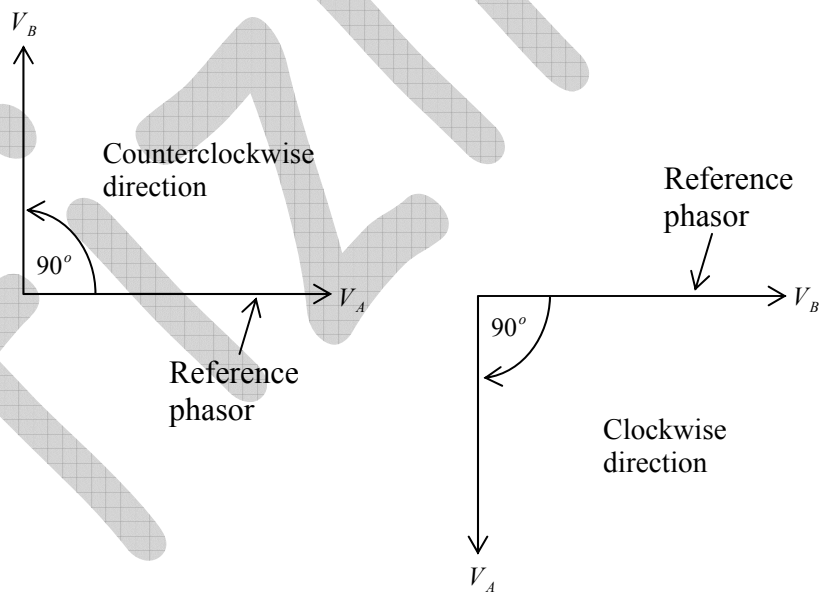


(*a*) Waveforms (*b*) Phasor diagram

Figure: Wave B leads wave A by a phase angle of 90°

9.2.5 Phasors

To compare phase angles or phases of alternating voltages and currents, it is more convenient to use phasor diagrams corresponding to the voltage and current waveforms. A phasor is a quantity that has magnitude and direction. The terms phasor and vector are used for quantities that have direction. However, a phasor quantity varies with time, while a vector quantity has direction in space. The length of the arrow in a phasor diagram indicates the magnitude of the alternating voltage. The angle of the arrow with respect to the horizontal axis indicates the phase angle. One waveform is chosen as the reference. Then the second waveform can be compared with the reference by means of the angle between the phasor arrows. For example, the phasor V_A represents the voltage wave A with a phase angle of 0° (figure *b* as shown above). The phasor V_B is vertical to show the phase angle of 90° with respect to phasor V_A, which is the reference. Since lead angles are shown in the counterclockwise direction from the reference phasor, V_B leads V_A by 90°.

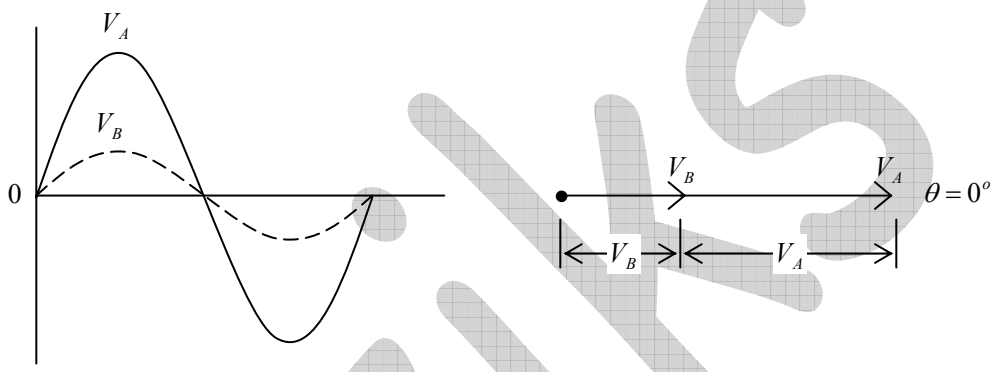


(*a*) V_B leads V_A by 90° (*b*) V_A lags V_B by 90°

Figure: Leading and lagging phase angles

Generally, the reference phasor is horizontal corresponding to 0°. If V_B were shown as the reference, V_A would have to be 90° clockwise in order to have the same phase angle. In this case V_A lags V_B by 90°. There is no fundamental difference between V_B leading V_A by 90° (figure *a* as shown below) or V_A lagging V_B by 90° (figure *b* as shown above).

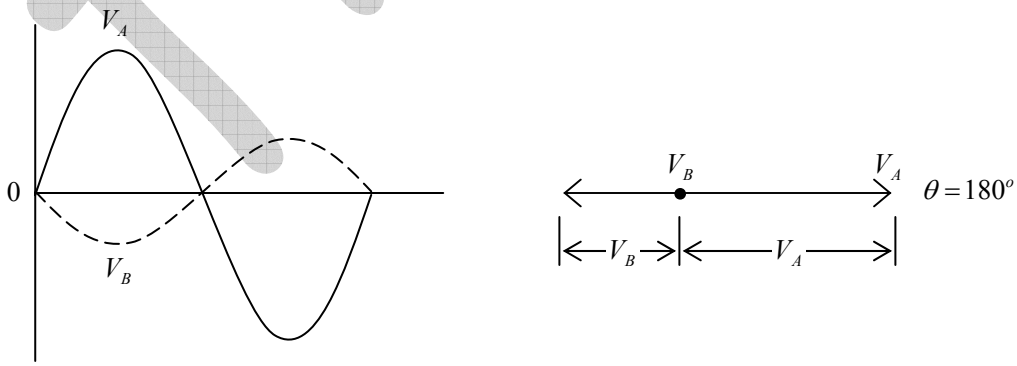
When two waves are in phase (figure *a* as shown below), the phase angle is zero. Then the amplitudes add (figure *b* as shown below).



(*a*) Waveforms (*b*) Phasor diagram

Figure: Two waves in phase with angle of 0^o

When two waves are exactly out of phase (figure *a* as shown below), the phase angle is 180°. Their amplitudes are opposing (figure *b* as shown below). Equal values of opposite phase cancel each other.



(*a*) Waveforms (*b*) Phasor diagram

Figure: Two waves in opposite phase with angle of 180^o

9.2.6 Characteristic Value of Voltage and Current

Since an ac sine wave voltage or current has many instantaneous values throughout the cycle, it is convenient to specify magnitudes for comparing one wave with another. The peak, average or root-mean-square (rms) value can be specified. These values apply to current or voltage.

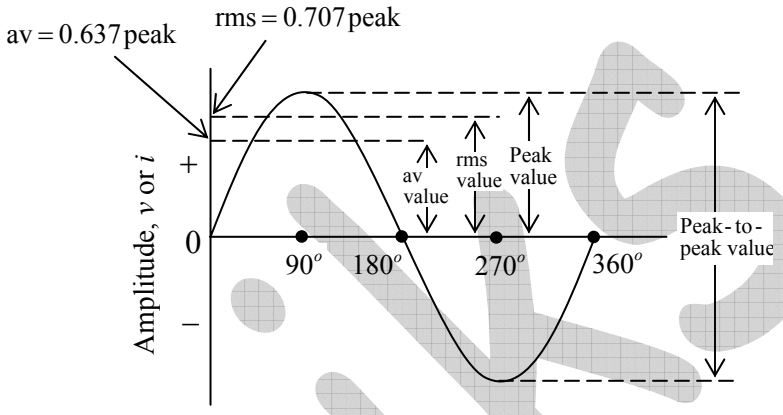


Figure: Amplitude values for ac sine wave

The *peak value* is the maximum value V_M or I_M. It applies to either the positive or negative peak. The *peak-to-peak* (p-p) value may be specified and is double the peak value when the positive and negative peaks are symmetrical.

The *average value* is the arithmetic average of all values in a sine wave for 1 half-cycle. The half-cycle is used for the average because over a full cycle the average value is zero.

$$\text{Average value} = 0.637 \times \text{peak value}$$

or
$$V_{av} = 0.637\, V_M$$
$$I_{av} = 0.637\, I_M$$

The *root-mean-square value* or *effective value* is 0.707 times the peak value.

$$\text{RMS value} = 0.707 \times \text{peak value}$$

or
$$V_{rms} = 0.707\, V_M$$
$$I_{rms} = 0.707\, I_M$$

The rms value of an alternating sine wave corresponds to the same amount of direct current or voltage in heating power. An alternating voltage with an rms value of 115 V, for example, is just as effective in heating the filament of a light bulb as 115 V from a steady dc voltage source. For this reason, the rms value is also called the effective value.

Unless indicated otherwise, all sine wave ac measurements are given in rms values. The letters V and I are used to denote rms voltage and current. For instance, V = 220 V (an ac power line voltage) is understood to mean 220 V rms.

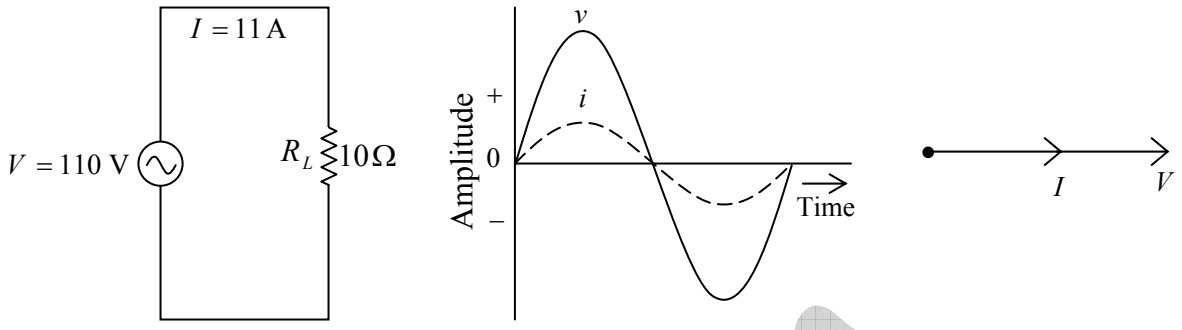
Use Table given below as a convenient way to convert from one characteristic value to another.

Table: Conversion Table for AC Sine Wave Voltage and Current

Multiply the Value	**By**	**To Get the Value**
Peak	2	Peak-to-peak
Peak-to-peak	0.5	Peak
Peak	0.637	Average
Average	1.570	Peak
Peak	0.707	RMS (effective)
RMS (effective)	1.414	Peak
Average	1.110	RMS (effective)
RMS (effective)	0.901	Average

9.2.7 Resistance in AC Circuits

In an ac circuit with only resistance, the current variations are in phase with the applied voltage (as shown in figure below). This in-phase relationship between V and I means that such an ac circuit can be analyzed by the same methods used for dc circuits. Therefore, Ohm's laws for dc circuits are applicable also to ac circuits with resistance only. The calculations in ac circuits are generally in rms values, unless otherwise specified. For the series circuit, $I = V/R = 110/10 = 11$ A. The rms power dissipation is $P = I^2R = 11^2(10) = 1210$ W.



(a) Schematic diagram (b) Waveforms of *i* and *c* (c) Phasor diagram, *I* in phase with *V*

Figure: AC circuit with only resistance

9.3 Inductive Circuits

9.3.1 Introduction

The ability of a conductor to induce voltage in itself when the current changes is its *self-inductance* or simply *inductance*. The symbol for inductance is L, and its unit is the henry (H). One henry is the amount of inductance that permits one volt to be induced when the current changes at the rate of one ampere per second. The formula for inductance is

$$L = \frac{v_L}{\Delta i/\Delta t}$$

where L = inductance, H

v_L = induced voltage across the coil, V

$\Delta i/\Delta t$ = rate of change of current, A/s

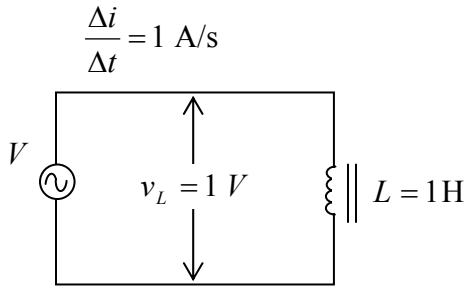


Figure: The inductance of a coil is 1H when a change of 1 A/s induces 1V across the coil

The self-induced voltage v_L from) is $v_L = L\dfrac{\Delta i}{\Delta t}$.

When the current in a conductor or coil changes, the varying flux can cut across any other conductor or coil located nearby, thus inducing voltages in both. A varying current in L_1, therefore, induces voltages across L_1 and across L_2 (as shown in figure below). When the induced voltage v_{L2} produces current in L_2, its varying magnetic field induces voltage in L_1. Hence, the two coils L_1 and L_2 have mutual inductance because current change in one coil can induce voltage in the other. The unit of mutual inductance is the henry, and the symbol is L_M. Two coils have L_M of 1H when a current change of 1 A/s in one coil induces 1 V in the other coil.

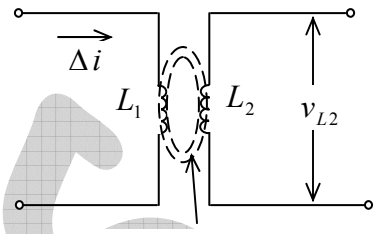


Figure: Mutual inductance Between L_1 and L_2

The schematic symbol for two coils with mutual inductance is shown in figure below.

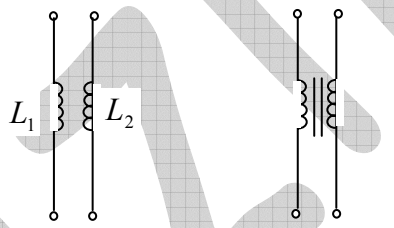


Figure: Schematic symbols for two coils with mutual inductance

9.3.2 Inductive Reactance

Inductive reactance X_L is the opposition to ac current due to the inductance in the circuit. The unit of inductive reactance is the ohm. The formula for inductive reactance is

$$X_L = 2\pi fL$$

Since $2\pi = 2(3.14) = 6.28$, $\quad X_L = 6.28\, fL$

where X_L = inductive reactance, Ω, $\quad f$ = frequency , Hz, $\quad L$ = inductance, H

If any two quantities are known the third can be found

$$L = \frac{X_L}{6.28\,f} \quad \text{and} \quad f = \frac{X_L}{6.28\,L}$$

In a circuit containing only inductance (as shown in figure below, Ohm's law can be used to find current and voltage by substituting X_L for R.

$$I_L = \frac{V_L}{X_L}$$

$$X_L = \frac{V_L}{I_L}$$

$$V_L = I_L X_L$$

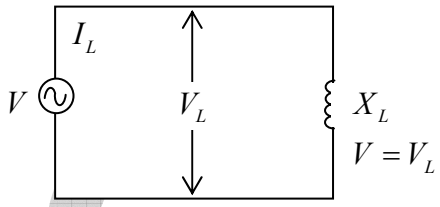


Figure: Circuit with only X_L

where I_L = current through the inductance, A

V_L = voltage across the inductance, V

X_L = inductive reactance, Ω

9.3.3 Inductors in Series or Parallel

If inductors are spaced sufficiently far apart so that they do not interact electromagnetically with each other; their values can be combined just like resistors when connected together. If a number of inductors are connected in series (as shown in figure below), the total inductance L_T is the sum of the individual inductances, or

Series: $L_T = L_1 + L_2 + L_3 + \cdots + L_n$

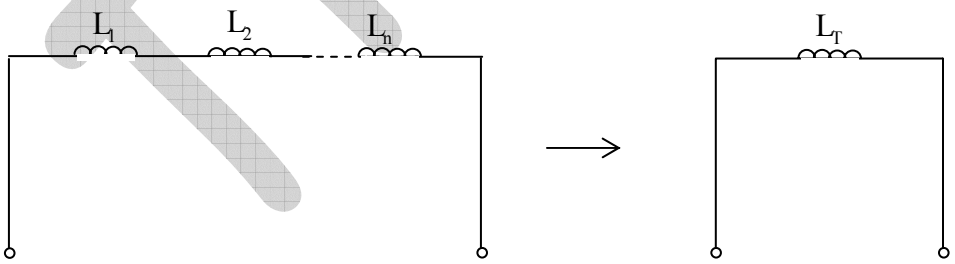


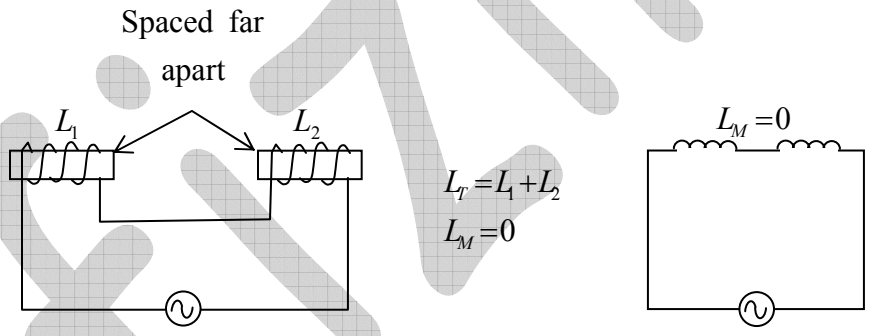
Figure: Inductances in series without mutual coupling

If two series-connected coils are spaced close together so that their magnetic field lines interlink, their mutual inductance will have an effect on the circuit. In that case the total inductance is

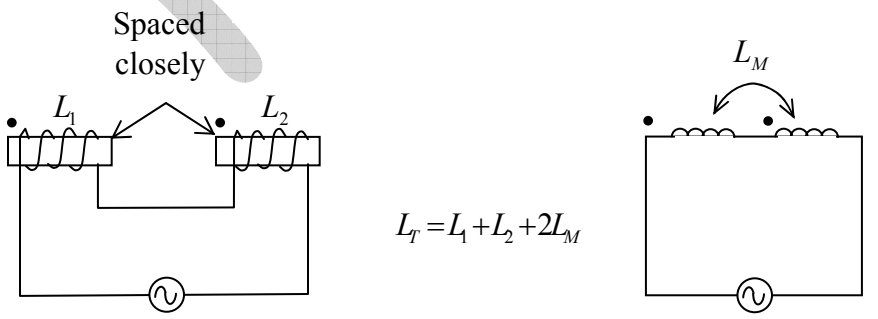
$$L_T = L_1 + L_2 \pm 2L_M$$

where L_M is the mutual inductance between the coils. The plus (+) sign is used if the coils are arranged in series-aiding form, while the minus (-) sign is used if the coils are connected in series-opposing form. Series aiding means that the common current produces the same direction of magnetic field for the two coils. The series-opposing connection results in opposite fields.

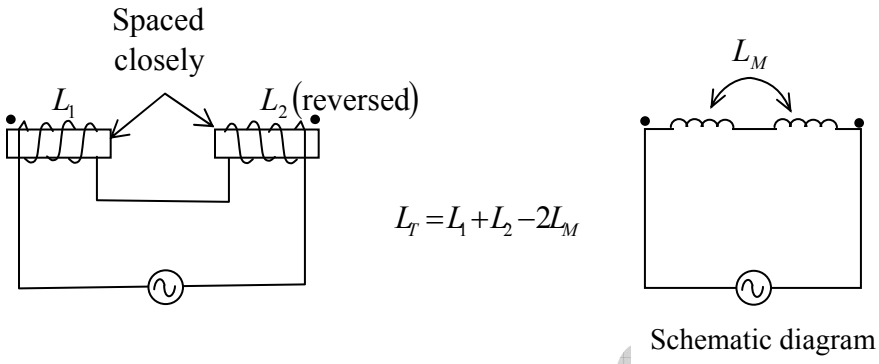
Three different arrangements for coils L_1 and L_2 are shown both pictorially and schematically as shown in figure below. In figue *a* coils are spaced too far apart to interact electromagnetically. There is no mutual inductance, so L_M is zero. The total inductance is $L_T = L_1 + L_2$. In figue *b*, the coils are spaced close together and have windings in the same direction, as indicated by the dots. The coils are series-aiding, so $L_T = L_1 + L_2 + 2L_M$. In figure *c*, the coil windings are in the opposite direction, so the coils are series-opposing, and $L_T = L_1 + L_2 - 2L_M$.



(a) No mutual inductance



(b) Series-aiding



(c) Series-opposing

Figure: L_1 and L_2 in series with mutual coupling L_M

The large dots above the coil (figure *b* and *c*) are used to indicate the polarity of the windings without having to show the actual physical construction. Coils with dots at the same end (figure *b*) have the same polarity or same direction of winding. When current enters the dotted ends for L_1 and L_2, their fields are aiding and L_M has the same sense as L.

If inductors are spaced sufficiently far apart so that their mutual inductance is negligible $(L_M = 0)$, the rules for combining inductors in parallel are the same as for resistors. If a number of inductors are connected in parallel (as shown in figure below), their total inductance L_T is

Parallel: $$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3} + \cdots + \frac{1}{L_n}$$

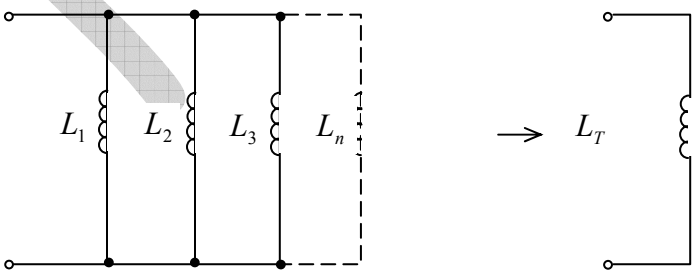


Figure: Inductances in parallel without mutual coupling

The total inductance of two coils connected in parallel is

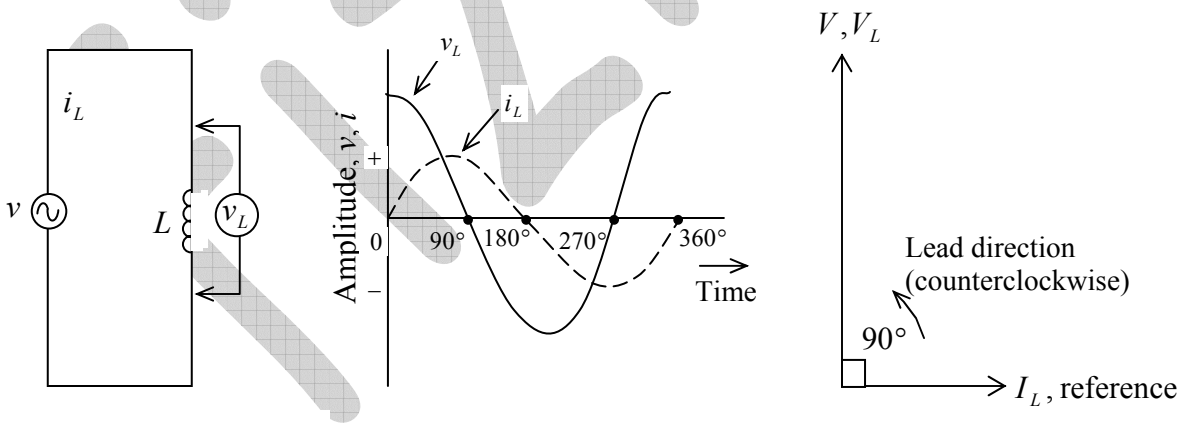
Parallel: $$L_T = \frac{L_1 L_2}{L_1 + L_2}$$

All inductances must be given as the same units. The shortcuts for calculating parallel R can be used with parallel L. For example, if two 8-mH inductors are in parallel, the total inductance is $L_T = L/n = 8/2 = 4\,\text{mH}$.

9.3.4 Inductive Circuits

9.3.4.1 Inductance Only

If an ac voltage v is applied across a circuit having only inductance (figure *a*), the resulting ac current through the inductance, i_L, will lag the voltage across the inductance, v_L, by 90° (figure *b* and *c*). Voltage v and v_L are the same because the total applied voltage is dropped only across the inductance. Both i_L and v_L are sine waves with the same frequency. Lowercase letters such as i and v indicate instantaneous values; capital letters such as I and V show dc or ac rms values.

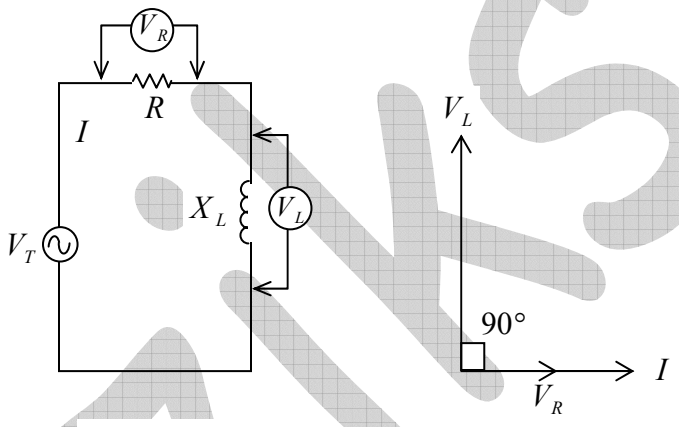


(a) Schematic diagram (c) Time diagram (c) Phasor diagram

Figure: Circuit with L only

9.3.4.2 *RL* in Series

When a coil has series resistance (figure *a* shown below), the rms current I is limited by both X_L and R. I is the same in X_L and R since they are in series. The voltage drop across R is $V_R = IR$, and the voltage drop across X_L is $V_L = IX_L$. The current I through X_L must lag V_L by 90° because this is the phase angle between current through an inductance and its self-induced voltage (figure *b*). The current I through R and its IR voltage drop are in phase so the phase angle is $0°$.



(a) Circuit (c) Phasor diagram

Figure: R and X_L in series

To combine two waveforms out of phase, we add their equivalent phasors. The method is to add the tail of one phasor to the arrowhead of the other, using the angle to show their relative phase. The sum of the phasors is a resultant phasors from the start of one phasor to the end of the other phasor. Since V_R and V_L phasors form a right angle, the resultant phasor is the hypotenuse of a right triangle. From the geometry of a right triangle, the resultant is

$$V_T = \sqrt{V_R^2 + V_L^2}$$

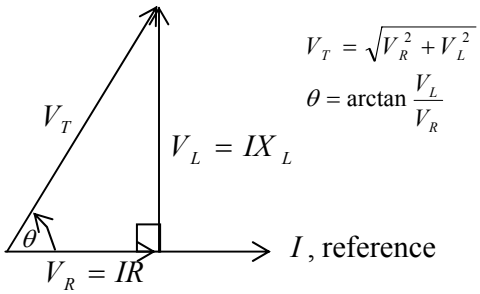


Figure: Phasor-voltage triangle

where the total voltage V_T is the phasor sum of the two voltages V_R and V_L that are 90° out of phase. All the voltages must be in the same units-rms values, peak values, or instantaneous values. For example, when V_T is an rms value, V_R and V_L also are rms values. Most of the ac calculations will be made in rms units.

The phase angle θ between V_T and V_R is $\tan\theta = \dfrac{V_L}{V_R}$ or $\theta = \arctan\dfrac{V_L}{V_R}$

Since V_R is in phase with I, θ is also the phase angle between V_T and I where I lags V_T.

Impedance in Series RL

The resultant of the phasor addition of R and X_L is called impedance. The symbol for impedance is Z. Impedance is the total opposition to the flow of current, expressed in ohms. The impedance triangle (as shown in figure) corresponds to the voltage triangle, but the common factor I cancels. The equations for impedance and phase angle are derived as follows:

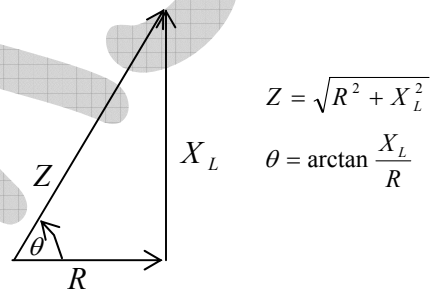


Figure: Phasor addition of R and X_L to find Z

$$V_T^2 = V_R^2 + V_L^2$$

$$(IZ)^2 = (IR)^2 + (IX_L)^2 \Rightarrow Z^2 = R^2 + X_L^2$$

$$Z = \sqrt{R^2 + X_L^2}$$

$$\tan\theta = \frac{X_L}{R} \text{ or } \theta = \arctan\frac{X_L}{R}$$

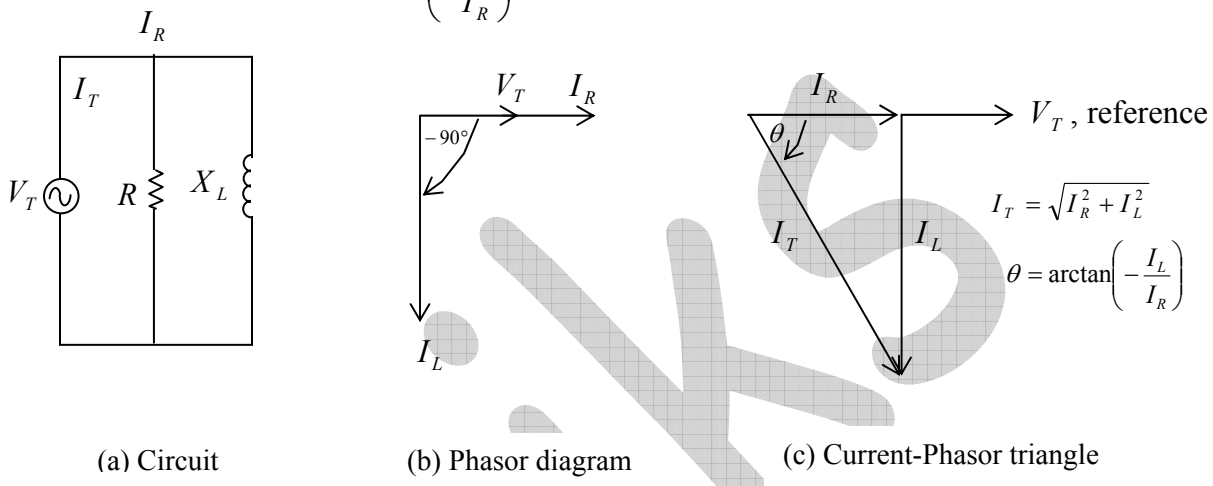
9.3.4.3 *RL* in Parallel

For parallel circuits with R and X_L (figure *a* shown below), the same applied voltage V_T is across R and X_L since both are in parallel with V_T. There is no phase difference between these voltages. Therefore V_T will be used as the reference phasor. The resistive branch current $I_R = V_T / R$ is in phase with V_T. The inductive branch current $I_L = V_T / X_L$ lags V_T by 90° (figure *b)* because the current in an inductance lags the voltage across it by 90°. The phasor sum of I_R and I_L equals the total line current I_T (figure c), or

$$I_T = \sqrt{I_R^2 + I_L^2}$$

$$\tan\theta = -\frac{I_L}{I_R}$$

$$\theta = \arctan\left(-\frac{I_L}{I_R}\right)$$



(a) Circuit (b) Phasor diagram (c) Current-Phasor triangle

Figure: R and X_L in parallel

Impedance in Parallel RL

For the general case of calculating the total impedance Z_T of R and X_L in parallel, assume any number for the applied voltage V_T because in the calculation of Z_T in terms of the branch currents the value of V_T cancels. A convenient value to assume for V_T is the value of either R or X_L, whichever is the higher number. This is only one method among others for calculating Z_T.

9.3.5 Q of a Coil

The quality or merit Q of a coil is indicated by the equation

$$Q = \frac{X_L}{R_i} = \frac{6.28fL}{R_i}$$

where R_i is the internal resistance of the coil equal to the resistance of the wire in the coil (figure *a*). Q is a numerical value without any units since the ohms cancel in the ratio of

reactance to resistance. If the Q of a coil is 200, it means that the X_L of the coil is 200 times more than its R_i. The Q of a coil may range in the value from less than 10 for a low-Q coil up to 1000 for a very high Q coil. Radio frequency (RF) coils have a Q of about 30 to 300.

As an example, a coil with an X_L of $300\,\Omega$ and a R_i of $3\,\Omega$ has a Q of 300/3 =100.

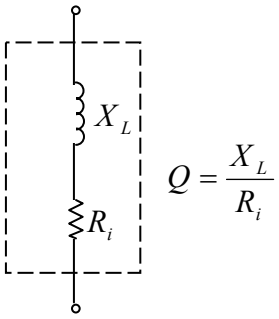


Figure: Schematic diagrm of Q of a coil. X_L and R_i are distributed uniformly over the length of the coil

9.3.6 Power in RL Circuit

In an ac circuit with inductive reactance, the line current I lags the applied voltage V. The real power P is equal to the voltage multiplied by one that portion of the line current which is in phase with the voltage. Therefore, Real power

$$\mathrm{P} = \mathrm{V}(\mathrm{I}\cos\theta) = \mathrm{VI}\cos\theta$$

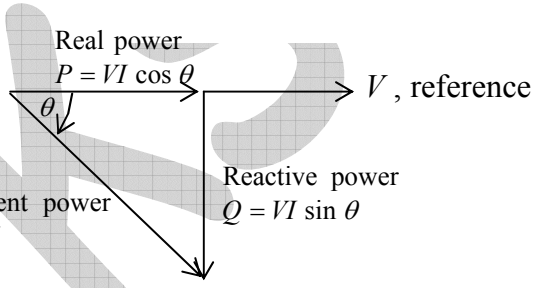


Figure: Power triangle for RL circuit

where θ is the phase angle between V and I, and $\cos\theta$ is the power factor (PF) of the circuit. Also, Real power $\mathrm{P} = I^2R$

where R is the total resistive component of the circuit.

Reactive power Q in voltamperes reactive (VAR), is expressed as follows:

Reactive power $Q = VI\sin\theta$

Apparent power S is the product of V x I. The unit is voltamperes (VA). In formula form,

Apparent power $S = VI$

In all the power formulas, the V and I are in rms values. The relationships of real, reactive and apparent power can be illustrated by the phasor diagram of power (as shown in above figure).

9.4 Capacitive Circuits

9.4.1 Capacitance

Electrically, capacitance is the ability to store an electric charge. Capacitance is equal to the amount of charge that can be stored in a capacitor divided by the voltage applied across the plates $C = \frac{Q}{V}$

where C = capacitance, F; Q = amount of charge, C; V = voltage, V

Above equation can be rewritten as follows: $Q = CV$ or $V = \frac{Q}{C}$.

The unit of capacitance is the farad (F). The farad is that capacitance that will store one coulomb of charge in the dielectric when the voltae applied across the capacitor terminals is one volt.

9.4.2 Capacitive Reactance

Capacitive reactance X_C is the opposition to the flow of ac current due to the capacitance in the circuit. The unit of capacitive reactance is the ohm. Capacitive reactance can be found by using the equation

$$X_C = \frac{1}{2\pi fC} = \frac{1}{6.28\, fC} = \frac{0.159}{fC}$$

where X_C = capacitive reactance, Ω ; f = frequency, Hz; C = capacitance, F

If any two quantities are known, the third can be found

$$C = \frac{0.159}{fX_C} \quad \text{and} \quad f = \frac{0.159}{CX_C}$$

Voltage and current in a circuit containing only capacitive reactance can be found using Ohm's law. However, in the case of a capacitive circuit, R is replaced by X_C.

$$V_C = I_C X_C \quad \text{or} \quad I_C = \frac{V_C}{X_C} \quad \text{or} \quad X_C = \frac{V_C}{I_C}$$

where I_C = current through the capacitor A; V_C = voltage across the capacitor, V

X_C = capacitive reactance, Ω

9.4.3 Capacitors in Series or Parallel

When capacitors are connected in series (as shown in figure below) the total capacitance C_T is

Series: $$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_n}$$

The total capacitance of two capacitors in series is

Series: $$C_T = \frac{C_1 C_2}{C_1 + C_2}$$

When n number of series capacitors have the same capacitance, $C_T = C/n$.

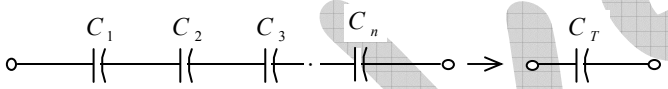


Figure: Capacitances in series

When capacitors are connected in parallel (as shown in figure below), the total capacitance C_T is the sum of the individual capacitances.

Parallel: $$C_T = C_1 + C_2 + C_3 + \cdots + C_n$$

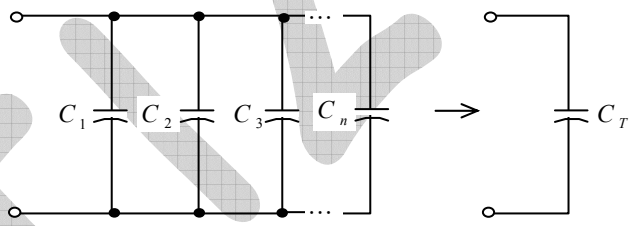
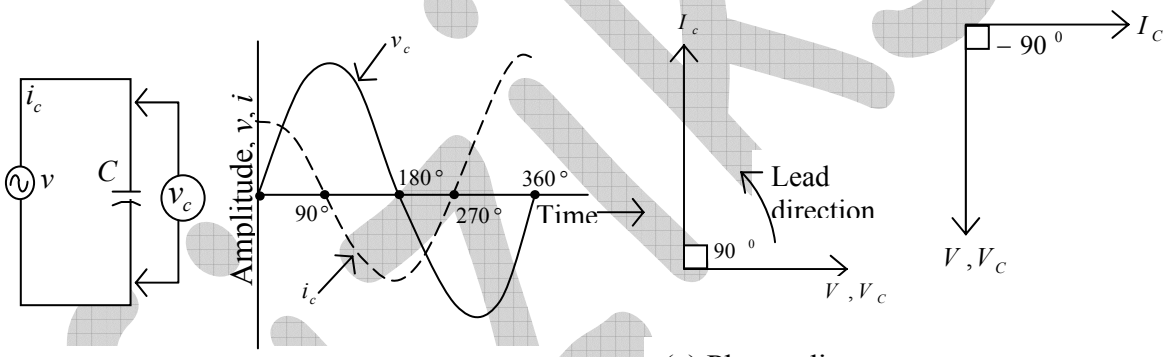


Figure: Capacitances in parallel

9.4.4 Capacitive Circuits

9.4.4.1 Capacitance Only

If an ac voltage v is applied across a circuit having only capacitance (figure *a*), the resulting ac current through the capacitance i_c, will lead the voltage across the capacitance, v_C, by 90° (figure *b* and *c*), (Quantities expressed as lowercase letters, i_c and v_C, indicate instantaneous values.) Voltages v and v_C are the same because they are parallel. Both i_c and v_C are sine waves with the same frequency. In series circuits, the current I_C is the horizontal phasor for reference (figure *d*) so the voltage V_C can be considered to lag I_C by $90°$.



(a) Schematic diagram

(b) Time diagram, i_c leads v_c by 90^0

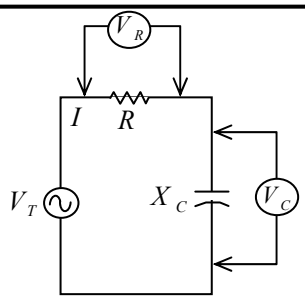
(c) Phasor diagram, V reference

(d) Phasor diagram, I_C reference

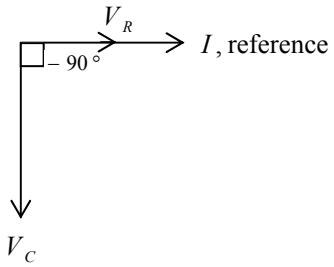
Figure: Circuit with *C* only

9.4.4.2 *RC* in Series

As with inductive circuits, the combination of resistance and capacitive reactance (figure *a* shown below) is called impedance. In a series circuit containing R and X_C, the same current I flows in X_C and R. The voltage drop across R is $V_R = IR$, and the voltage drop across X_C is $V_C = IX_C$. The voltage across X_C lags the current through X_C by $90°$ (figure *b* shown below). The voltage across R is in phase with I since resistance does not produce a phase shift.



(a) Circuit



(b) Phasor diagram

Figure: R and X_C in series

To find the total voltage V_T, we add phasors V_R and V_C. Since they form a right triangle (as shoen in figure below), $V_T = \sqrt{V_R^2 + V_C^2}$

Note that the IX_C phasor is downward, exactly opposite from an IX_L phasor, because of the opposite phase angle.

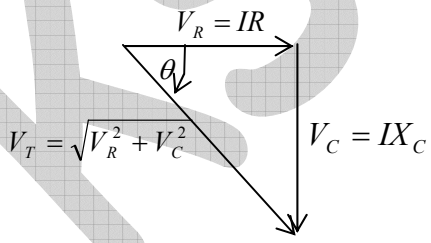


Figure: Voltage-triangle phasor

The phase angle θ between V_T and V_R is expressed according to the following equation:

$$\tan\theta = -\frac{V_C}{V_R} \qquad \text{or} \qquad \theta = \arctan\left(-\frac{V_C}{V_R}\right)$$

Impedance in Series RC

The voltage triangle corresponds to the impedance triangle because the common factor I in V_C and V_R cancels.

$$V_C = IX_C,\ V_R = IR,\ \tan\theta = -\frac{IX_C}{IR} = -\frac{X_C}{R}$$

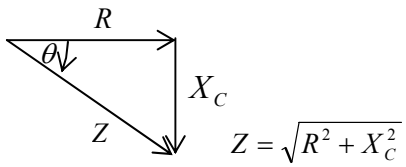


Figure: Series RC impedance triangle

Impedance Z is equal to the phasor sum of R and X_C

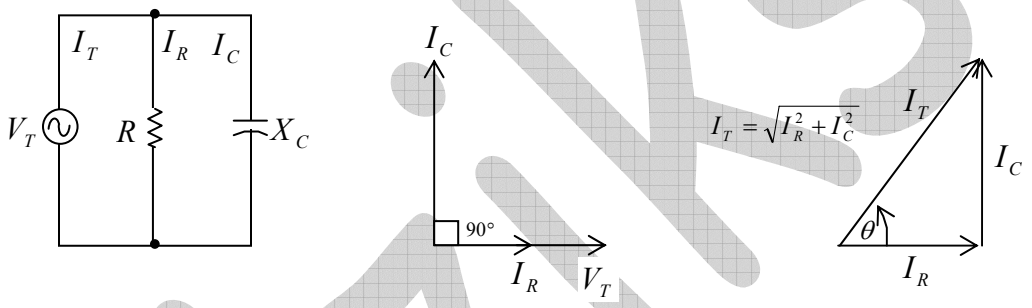
$$Z = \sqrt{R^2 + X_C^2}$$

The phase angle θ is

$$\theta = \arctan\left(-\frac{X_C}{R}\right)$$

9.4.4.3 *RC* in Parallel

In the *RC* parallel circuit (figure *a* shown below), the voltage is the same across the source, *R,* and X_C since they are all in parallel. Each branch has its individual current. The resistive branch current $I_R = V_T / R$ is in phase with V_T. The capacitive branch current $I_C = V_T / X_C$ leads V_T by 90° (figure *b* shown below). The phasor diagram has the source voltage V_T as the reference phasor because it is the same throughout the circuit. The total line current I_T equals the phasor sum of I_R and I_C (figure *c* shown below).



(a) Circuit (b) Phasor diagram (c) Current- triangle phasor

Figure: X_C and *R* in parallel

$$I_T = \sqrt{I_R^2 + I_C^2}, \quad \tan\theta = \frac{I_C}{I_R} \quad \text{or} \quad \theta = \arctan\frac{I_C}{I_R}$$

Impedance in Parallel RC

The impedance of a parallel circuit equals the total voltage V_T divided by the total current I_T.

$$Z_T = \frac{V_T}{I_T}$$

9.4.5 Power in *RC* Circuits

The power formulas given previously for *RL* circuits are equally applicable to *RC* circuits.

Real power $P = VI\cos\theta$

$$P = I^2R = \frac{V^2}{R}$$

Reactive power $Q = VI\sin\theta$

Apparent power $S = VI$

Note: Capacitance, like inductance, consumes no power. The only part of the circuit consuming power is the resistance.

Summary Table for Series and Parallel RC Circuits

X_C and R in Series	X_C and R in Parallel
I the same in X_C and R	V_T the same across X_C and R
$V_T = \sqrt{V_R^2 + V_C^2}$	$I_T = \sqrt{I_R^2 + I_C^2}$
$Z = \sqrt{R^2 + X_C^2} = \frac{V_T}{I}$	$Z_T = \frac{V_T}{I_T}$
V_C lags V_R by 90°	I_C lags I_R by 90°
$\theta = \arctan\left(-\frac{X_C}{R}\right)$	$\theta = \arctan\frac{I_C}{I_R}$

9.5 Single Phase Circuits

9.5.1 *RLC* in Series

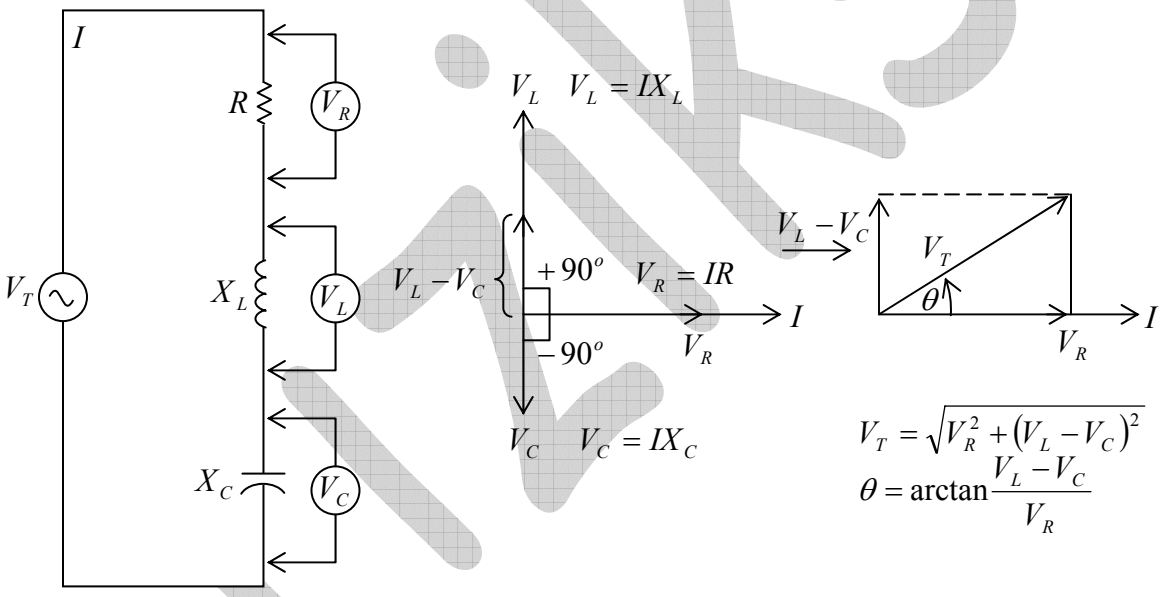
Current in a series circuit containing resistance, inductive reactance, and capacitive reactance (figure *a* shown below) is determined by the total impedance of the combination. The current I is the same in R, X_L, and X_C since they are in series. The voltage drop across each element is found by Ohm's law:

$$V_R = IR \qquad V_L = IX_L \qquad V_C = IX_C$$

Where V_R = voltage drop across the resistance, V

V_L = voltage drop across the inductance, V

V_C = voltage drop across the capacitance, V



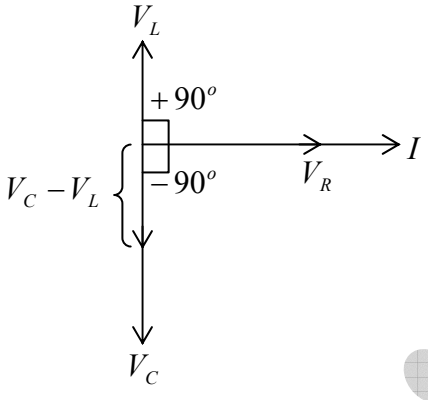
(*a*) Series RLC circuit diagram (*b*) Phasor diagram; $V_L > V_C$ (*c*) Voltage-phasor triangle, $V_L > V_C$

Figure: R, X_L, and X_C in series; $X_L > X_C$ for inductive circuit

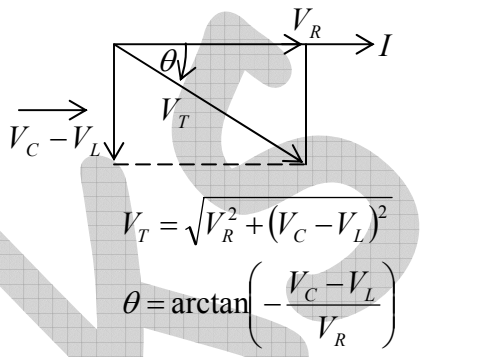
The voltage drop across the resistance is in phase with current through the resistance (figure *b* shown above). The voltage across the inductance leads the current through the capacitance by 90°. The voltage across the capacitance lags the current through the capacitance by 90° . Since V_L and V_C are exactly 180° out of phase and acting in exactly

opposite directions, they are added algebraically. When X_L is greater than X_C the circuit is inductive, V_L is greater than V_C and I lags V_T (figure *c* shown below).

When X_C is greater than X_L, the circuit is capacitive. Now V_C is greater than V_L so that I leads V_T (figure *shown* below)



(*a*) Phasor diagram; $V_C > V_L$



(*b*) Voltage-phasor triangle, $V_C > V_L$

Figure: R, X_L, and X_C in series; $X_C > X_L$ for capacitive circuit

When $X_L > X_C$, the voltage-phasor diagram shows that the total voltage V_T and phase angle are as follows:

$$V_T = \sqrt{V_R^2 + (V_L - V_C)^2} \quad \text{and} \quad \theta = \arctan\frac{V_L - V_C}{V_R}$$

When $X_C > X_L$

$$V_T = \sqrt{V_R^2 + (V_C - V_L)^2} \quad \text{and} \quad \theta = \arctan\left(\frac{V_C - V_L}{V_R}\right)$$

9.5.1.1 Impedance in Series *RLC*

Impedance Z is equal to the phasor sum of R, X_L, and X_C. In figure *a* shown below:

When $X_L > X_C,\ Z = \sqrt{R^2 + (X_L - X_C)^2}$

In figure *b* shown below:

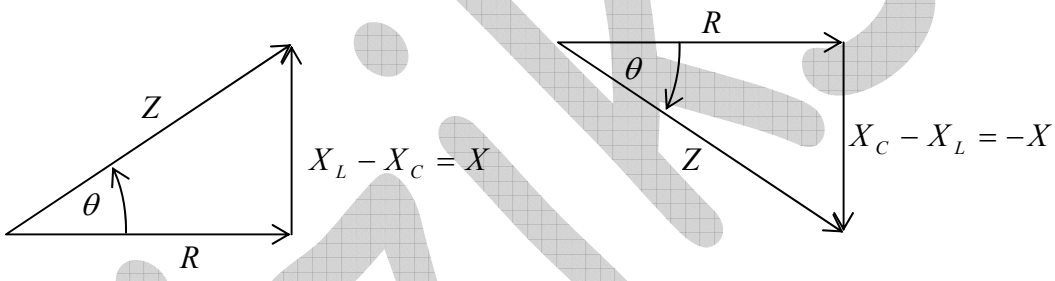
When $X_C > X_L,\ Z = \sqrt{R^2 + (X_C - X_L)^2}$

It is convenient to define net reactance X as

$$X = X_L - X_C$$

Then $Z = \sqrt{R^2 + X^2}$

for both inductive and capacitive *RLC* series circuits.



$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + X^2}$ $\quad$ $Z = \sqrt{R^2 + (X_C - X_L)^2} = \sqrt{R^2 + X^2}$

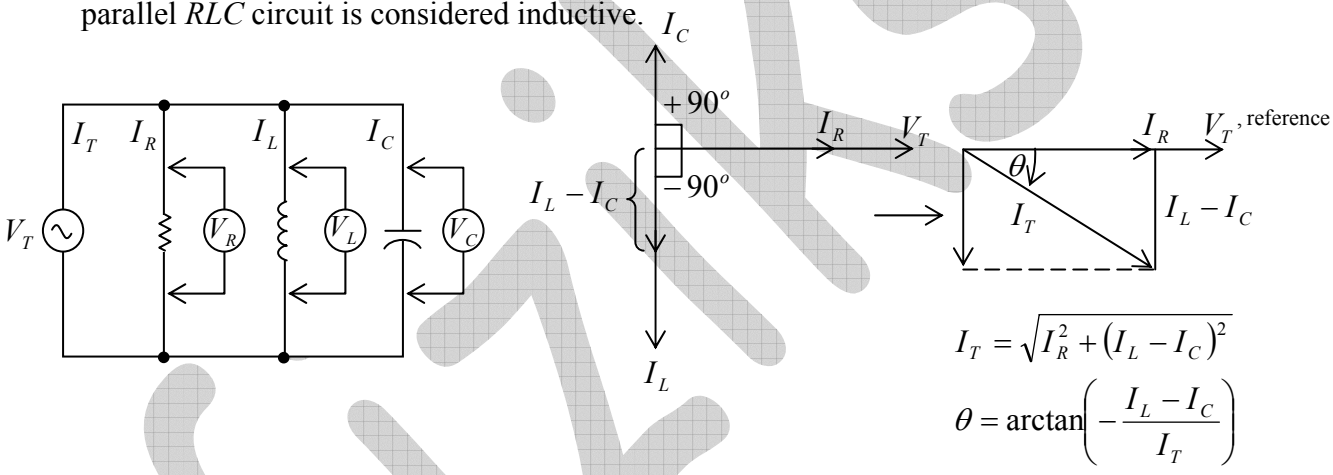
$\theta = \arctan\dfrac{X}{R}$ $\quad$ $\theta = \arctan\left(-\dfrac{X}{R}\right)$

$(a) X_L > X_C$ $\quad$ $(b) X_C > X_L$

Figure: Series *RLC* impedance-phasor triangle

9.5.2 *RLC* in Parallel

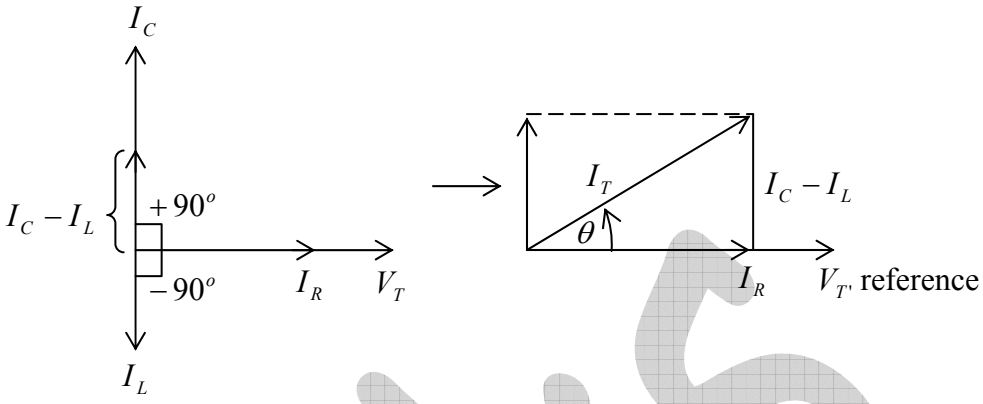
A three- branch parallel ac circuit (figure *a* shown below) has resistance in one branch, inductance in the second branch, and capacitance in the third branch. The voltage is the same across each parallel branch, so $V_T = V_R = V_L = V_C$. The applied voltage V_T is used as the reference line to measure phase angle θ. The total current I_T is the phasor sum of I_R, I_L, and I_C. The current in the resistance I_R is in phase with the applied voltage V_T (figure *b* shown below). The current in the inductance I_L lags the voltage V_T by 90°. The current in the capacitor I_C leads the voltage V_T by 90°. I_L and I_C are exactly 180° out of phase and thus acting in opposite directions. When $I_L > I_C$, I_T lags V_T (figure *c* shown below) so the parallel *RLC* circuit is considered inductive.



(*a*) Parallel RLC circuit diagram (*b*) Phasor diagram, $I_L > I_C$ (*c*) Current-phasor triangle, $I_L > I_C$

Figure: R, X_L, and X_C in parallel; $I_L > I_C$ for inductive circuit

If $I_C > I_L$, the current relationship and phasor triangle (figure shown below) show that I_T now leads V_T so this type of parallel RLC circuit is considered capacitive.



(a) Phasor diagram; $I_C > I_L$ (b) Current-phasor triangle, $I_C > I_L$

Figure: R, X_L, and X_C in parallel; $I_C > I_L$ for capacitive circuit

When $I_L > I_C$ the circuit is inductive and

$$I_T = \sqrt{I_R^2 + (I_L - I_C)^2} \quad \text{and} \quad \theta = \arctan\left(-\frac{I_L - I_C}{I_R}\right)$$

and when $I_C > I_L$, the circuit is capacitive and

$$I_T = \sqrt{I_R^2 + (I_C - I_L)^2} \quad \text{and} \quad \theta = \arctan\frac{I_C - I_L}{I_R}$$

Note: In a parallel RLC circuit, when $X_L > X_C$, the capacitive current will be greater than the inductive current and the circuit is capacitive. When $X_C > X_L$, the inductive current is greater than the capacitive current and the circuit is inductive. These relationships are opposite to those for a series RLC circuit.

9.5.2.1 Impedance in Parallel *RLC*

The total impedance Z_T of a parallel RLC circuit equals the total voltage V_T divided by the total current I_T.

$$Z_T = \frac{V_T}{I_T}$$

9.5.3 *RL* and *RC* in Parallel

Total current I_T for a circuit containing parallel branches of *RL* and *RC* (figure shown below) is the phasor sum of the branch currents I_1 and I_2. A convenient way to find I_T is to

(1) Add algebraically horizontal components of I_1 and I_2 with respect to the phasor reference V_T,

(2) Add algebraically the vertical components of I_1 and I_2, and

(3) Form a right triangle with these two sums as legs and calculate the value of the hypotenuse (I_T) and its angle to the horizontal.

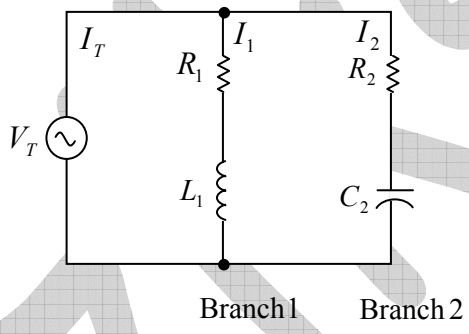


Figure: Parallel RL and RC branches

9.5.4 Power and Power Factor

The instantaneous power p is the product of the current i and the voltage v at that instant of time t.

$$p = vi$$

When v and i are either positive or both negative, their product p is positive. Therefore, power is being expended throughout the cycle (figure *a* shown below). If v is negative while i is positive during any part of the cycle (figure *b* shown below), or if i is negative while v is positive, their product will be negative. This "negative power" is not available for work; it is power returned to the line.

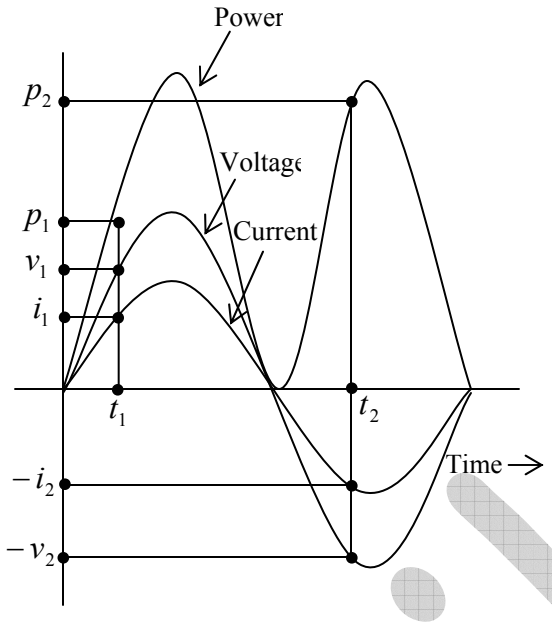


Figure (a): Power-time diagram when voltage and current are in phase

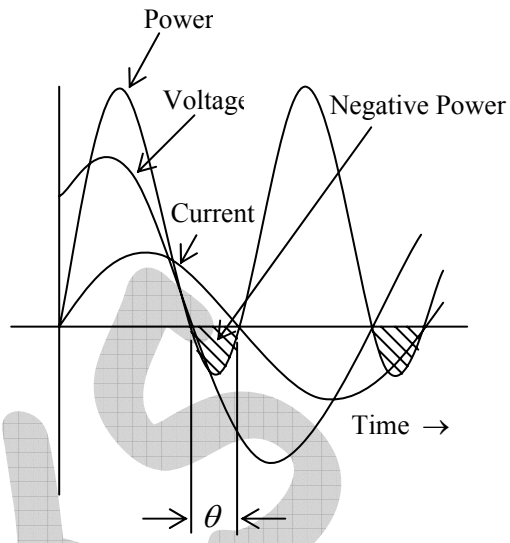


Figure (b): Power-time diagram in series *RL* circuit where current lags voltage by phase angle θ

The product of the voltage across the resistance and the current through the resistance is always positive and is called *real power*. Real power can be considered as resistive power that is dissipated as heat. Since the voltage across a reactance is always 90° out phase with the current through the reactance, the product $p_x = v_x i_x$ is always negative. This product is called reactive power and is due to the reactance of a circuit. Similarly, the product of the line voltage and the line current is known as apparent power.

Real power, *reactive power*, and *apparent power* can be represented by a right triangle (figure *a* shown below). From this triangle the power formulas are:

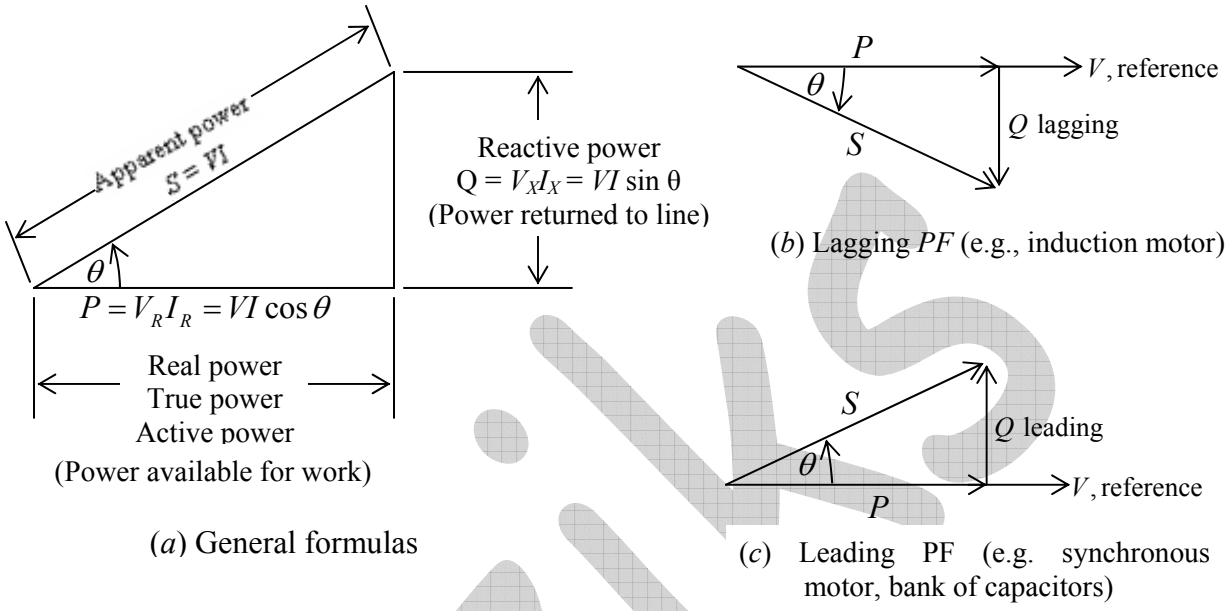
Real power $P = V_R I_R = VI\cos\theta$, W

or $\quad P = I^2 R \quad \Rightarrow \quad P = \dfrac{V^2}{R}, W$

Reactive power $Q = V_X I_X = VI\sin\theta$, VAR

Apparent power $S = VI$, VA

With line voltage V as reference phasor, in an inductive circuit, S lags P (figure b shown below); while in a capacitive circuit, S leads P (figure c shown below).



(*a*) General formulas

(*b*) Lagging *PF* (e.g., induction motor)

(*c*) Leading PF (e.g. synchronous motor, bank of capacitors)

Figure: Power triangle

The ratio of real power to apparent power, called the *power factor* (*PF*), is

Real power

$$PF = \frac{\text{Real power}}{\text{apparent power}} = \frac{V_R I_R}{VI} = \frac{VI\cos\theta}{VI} = \cos\theta$$

Also $$PF = \cos\theta = \frac{P}{VI}$$

The $\cos\theta$ of a circuit is the power factor, *PF*, of the circuit. The power factor determines what portion of apparent power is real power and can vary from 1 when the phase angle θ is 0°, to 0 when θ is 90°. When $\theta = 0^\circ$, $P = VI$, the formula for voltage and current of circuit in phase. When $\theta = 90^\circ$, $P = VI \times 0 = 0$, indicating that no power is being expended or consumed.

A circuit in which the current lags the voltage (i.e. an inductive circuit) is said to have a lagging *PF*; a circuit in which the current leads the voltage (i.e., a capacitive circuit) is said to have a leading *PF*.

Power factor is expressed as a decimal or as a percentage. A power factor of 0.7 is the same as a power factor of 70 percent. At unity (PF = 1, or 100 percent), the current and voltage are in phase. A 70 percent PF means that the device uses only 70 percent of the voltampere input. It is desirable to design circuits that have a high *PF* since such circuits make the most efficient use of the current delivered to the load.

When we state that a motor draws 10 kVA (1 kVA = 1000 VA) from a power line, we recognize that this is the apparent power taken by the motor. Kilovoltamperes always refers to the apparent power. Similarly, when we say a motor draws 10 kW, we mean that the real power taken by the motor is 10 kW.

Power Factor Correction

In order to make the most efficient use of the current delivered to a load, we desire a high *PF* or a *PF* that approaches unity. A low *PF* is generally due to the large inductive loads, such as induction motors, which take a lagging current. In order to correct this low *PF*, it is necessary to bring the current as closely in phase with the voltage as possible. That is, the phase angle θ is made as small as possible. This is usually done by placing a capacitive load, which produces a leading current, in parallel with the inductive load.

9.6 Series and Parallel Resonance

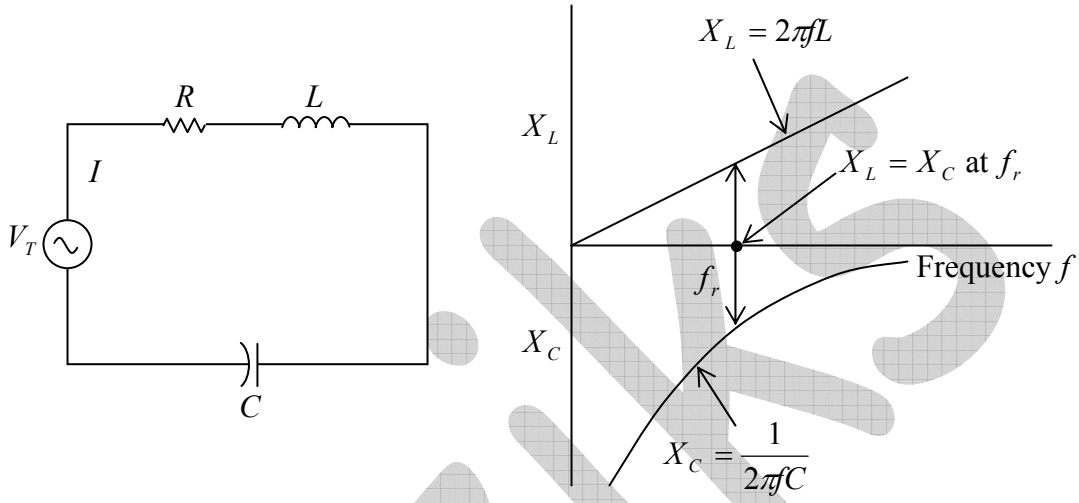
We have observed that in many circuits' inductors and capacitors are connected in series or in parallel. Such circuits are often referred to as *RLC* circuits. One of the most important characteristics of a *RLC* circuit is that it can be made to respond most effectively to a single given frequency. When operated in this condition, the circuit is said to be in *resonance* with or *resonant* to the operating frequency.

A series or a parallel *RLC* circuit that is operated at resonance has certain properties that allow it to respond selectively to certain frequencies while rejecting others. A circuit operated to provide frequency selectivity is called a tuned circuit. Tuned circuits are used in impedance matching, bandpass filters, and oscillators.

9.6.1 Series Resonance

The *RLC* series circuit (figure *a* shown below) has an impedance $Z = \sqrt{R^2 + (X_L - X_C)^2}$.

The circuit is at resonance when the inductive reactance X_L is equal to the capacitive reactance X_C (figure *b* shown below).



(*a*) Schematic diagram (*b*) $X_L = X_C$ but are 180° out of phase at f_r

Figure: Series resonance for *RLC* circuit at resonant frequency f_r

At resonance $X_L = X_C$

Where $X_L = 2\pi f L$ and $X_C = \dfrac{1}{2\pi f C}$

Then at resonance, $2\pi f L = \dfrac{1}{2\pi f C} \Rightarrow f^2 = \dfrac{1}{(2\pi)^2 LC}$

$$f = f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{0.159}{\sqrt{LC}} \quad \ldots\ldots\ldots\ldots (1)$$

where f_r = resonance frequency, Hz; L = inductance, *H*; C = capacitance, *F*

For any *LC* product [equation (1)] there is only one resonant frequency. Thus, various combinations of *L* and *C* may be used to achieve resonance if the *LC* product remains the

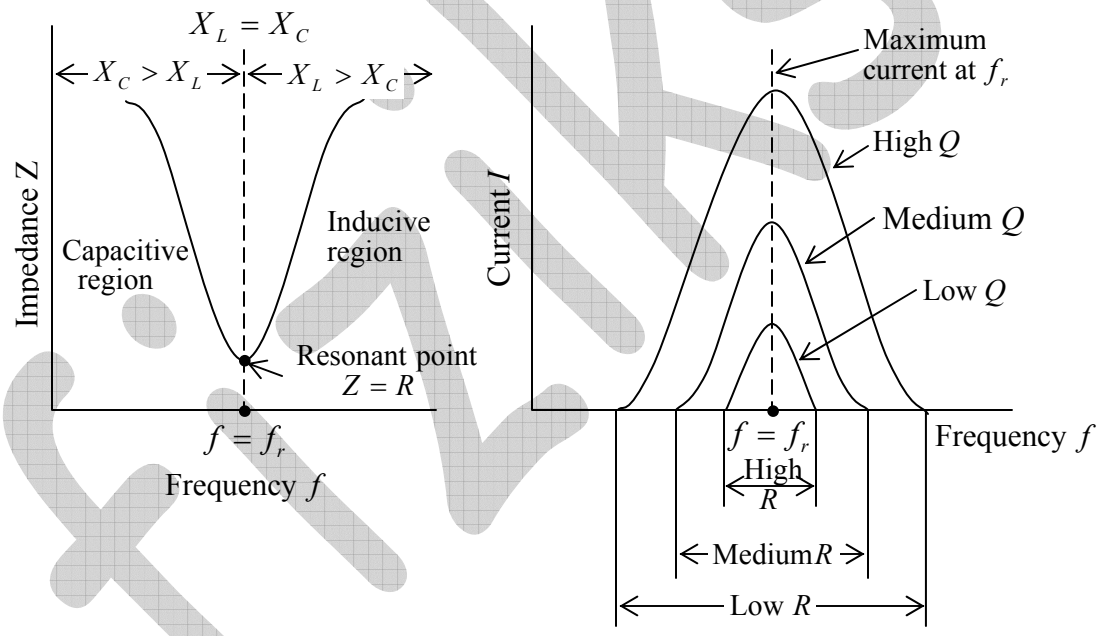
same. Equation (1) may be solved for L or C to find the inductance or capacitance needed to from a series resonant circuit at a given frequency.

$$L = \frac{1}{4\pi^2 f_r^2 C} = \frac{0.0254}{f_r^2} \text{ and } C = \frac{1}{4\pi^2 f_r^2 L} = \frac{0.0254}{f_r^2 L}$$

Since $X_L = X_C,\ X_L - X_C = 0$ so that

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2} = R$$

Since the impedance at resonance Z equals the resistance R, the impedance is a minimum. With minimum impedance, the circuit has maximum current determined by $I = V/R$. The resonant circuit has a phase angle equal to 0^0 so that the power factor is unity.



(a) Impedance response curve (b) Current response curve

Figure: Characteristics of series RLC circuit at resonance

At frequencies below the resonant frequency (figure a shown above). X_C is greater than X_L so the circuit consists of resistance and capacitive reactance. However, at frequencies above the resonant frequency, X_L is greater than X_C so that circuit consists of resistance and inductive reactance. At resonance, maximum current is produced for different values of resistance (figure b shown above). With a low resistance, maximum

current increases sharply toward and decreases sharply from its maximum current as the circuit is tuned to and away from the resonant frequency. This condition where the curve is narrow at the resonant frequency provides good selectivity. With an increase of resistance, the curve broadens so that selectivity is less.

9.6.1.1 *Q* of Series Circuit

The degree to which a series-tuned circuit is selective is proportional to the ratio of its inductive reactance to its resistance. This ratio $\frac{X_L}{R}$ is known as the Q of the circuit and is expressed as follows: $Q = \frac{X_L}{R}$

where Q = quality factor or figure of merit

X_L = inductive reactance, Ω

R = resistance, Ω

The lower the resistance, the higher the value of Q, the higher the Q the sharper and more selective is the resonant curve. Q has the same value if calculated with X_C instead of X_L since they are equal at resonance. Q = 150 is a high Q. Typical values are 50 to 250. Less than 10 is a low Q; more than 300 is a very high Q.

The Q of the circuit of the circuit is generally considered in terms of X_L since the coil has the series resistance of the circuit. In this case, the Q of the coil and the Q of the series resonant circuit are the same. If extra resistance is added, the Q of the circuit will be less than the Q of the coil. The highest possible Q for the circuit is the Q of the coil.

The Q of the resonant circuit can be considered a magnification factor that determines how much the voltage across L or C is increased by the resonant rise of current in a series circuit.

$$V_L = V_C = QV_T$$

9.6.2 Parallel Resonance

9.6.2.1 Pure Parallel *LC* circuit

In the pure *LC* parallel-tuned circuit (that is, one in which there is no resistance), the coil and capacitor are placed in parallel and the applied voltage V_T appears across these circuit components (figure shown below). In this parallel-tuned circuit, as in the series-tuned circuit, resonance occurs when the inductive reactance is equal to the capacitive reactance.

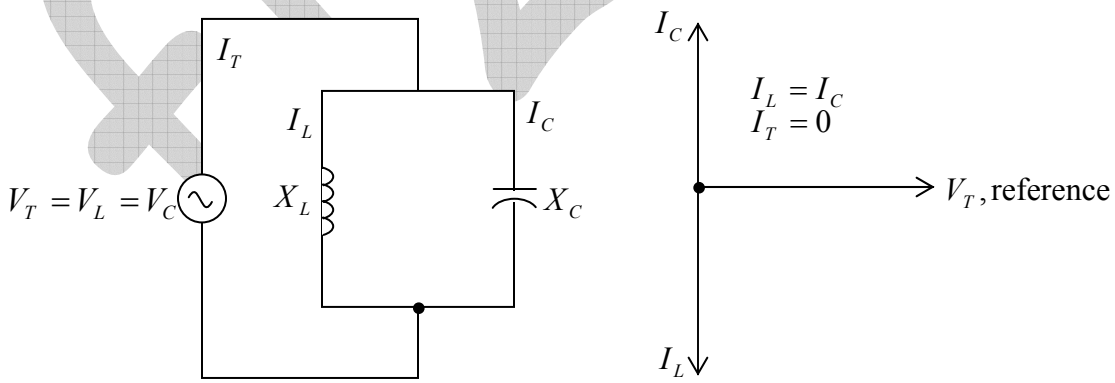
$$X_L = X_C$$

Because the applied voltage is common to both branches,

$$V_L = V_C$$

So that $$\frac{V_L}{X_L} = \frac{V_C}{X_C}, \qquad I_L = I_C$$

The current in the inductive branch I_L equals the current in the capacitive branch I_C. I_L lags the applied voltage V_T by 90°, while I_C leads the voltage by 90° (figure *b* shown below). Since the phasor currents I_L and I_C are equal and out of phase by 180°, their vector sum is zero so that the total current I_T is zero. Under this condition the impedance of the circuit at the resonant frequency must be infinite in value.



(*a*) Schematic diagram (*b*) Current-phasor diagram at resonance

Figure: Pure *LC* parallel circuit

The formula for the resonant frequency of a pure LC parallel-tuned circuit is the same as that for a series circuit.

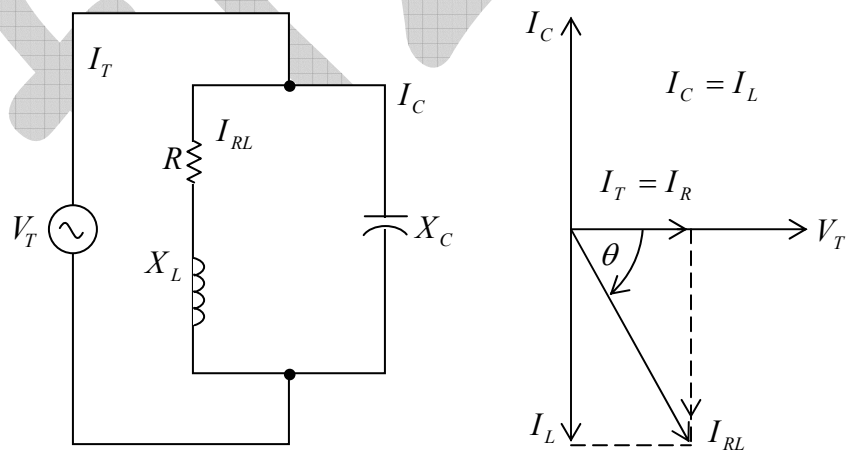
$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{0.159}{\sqrt{LC}}$$

If the resonant frequency is known, then the inductance or capacitance for a parallel resonant LC circuit can be found by formulas.

$$L = \frac{0.0254}{f_r^2 C} \quad \text{and} \quad C = \frac{0.0254}{f_r^2 L}$$

9.6.2.2 Practical Parallel *LC* Circuit

In a practical LC parallel-tuned circuit (figure *a* shown below), there is some resistance, most of which is due to the resistance of the inductor wire. The resonant frequency of a parallel circuit also is defined as that frequency at which the parallel circuit acts as a pure resistance. Therefore, the line current I_T must be in phase with the applied voltage V_T (unity power factor) (figure *b* shown below). This means that the out-of-phase or quadrature component of the current through the inductive branch I_L must be equal to the current through the capacitive branch I_C; and the total line current I_T equals the in-phase component of the current through the inductive branch, or $I_T = I_r$ (figure *b* shown below). Since the impedance is maximum, I_T is minimum.



(*a*) Schematic diagram (*b*) Current-phasor diagram at resonance

Figure: Practical parallel LC circuit

The resonant frequency for the circuit (figure *a* shown above) is

$$f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}} \quad \ldots\ldots\ldots\ldots (2)$$

where f_r = resonant frequency, Hz; L = inductance, H; C = capacitance, F; R = resistance, Ω.

If the Q of the coil is high, say greater than 10, or the term $1/LC >> R^2/L^2$, then, for practical purposes, the term R^2/L^2 can be disregarded. The result is that equation (2) becomes equation (1), the resonant frequency formula for series resonance.

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

the total impedance at resonance of the practical LC parallel circuit is

$$Z_T = \frac{1}{RC}$$

In terms of quality factor Q, Z_T at resonance can also be found by

$$Z_T = X_L Q = 2\pi f_r L Q \quad \text{or} \quad Z_T = \frac{Q}{2\pi f_r C}$$

The impedance Z_T of a practical parallel circuit is maximum at the resonant frequency and decreases at frequencies below and above the resonant frequency (figure *a* shown below). An increase in resistance decreases the impedance and causes the impedance to vary less "sharply" as the circuit is tuned over a band of frequencies below and above the resonant frequency (figure *b* shown below). At frequencies below resonance, $X_C > X_L$ and $I_L > I_C$ so that the parallel-tuned circuit is inductive (figure *a* and *c* shown below). At frequencies above resonance, the reverse condition is true, $X_L > X_C$ and $I_C > I_L$, so that now the circuit is capacitive (figure *a* and *c* shown below c). Since the impedance Z_T is maximum at parallel resonance, I_T is minimum (figure *c* shown below).

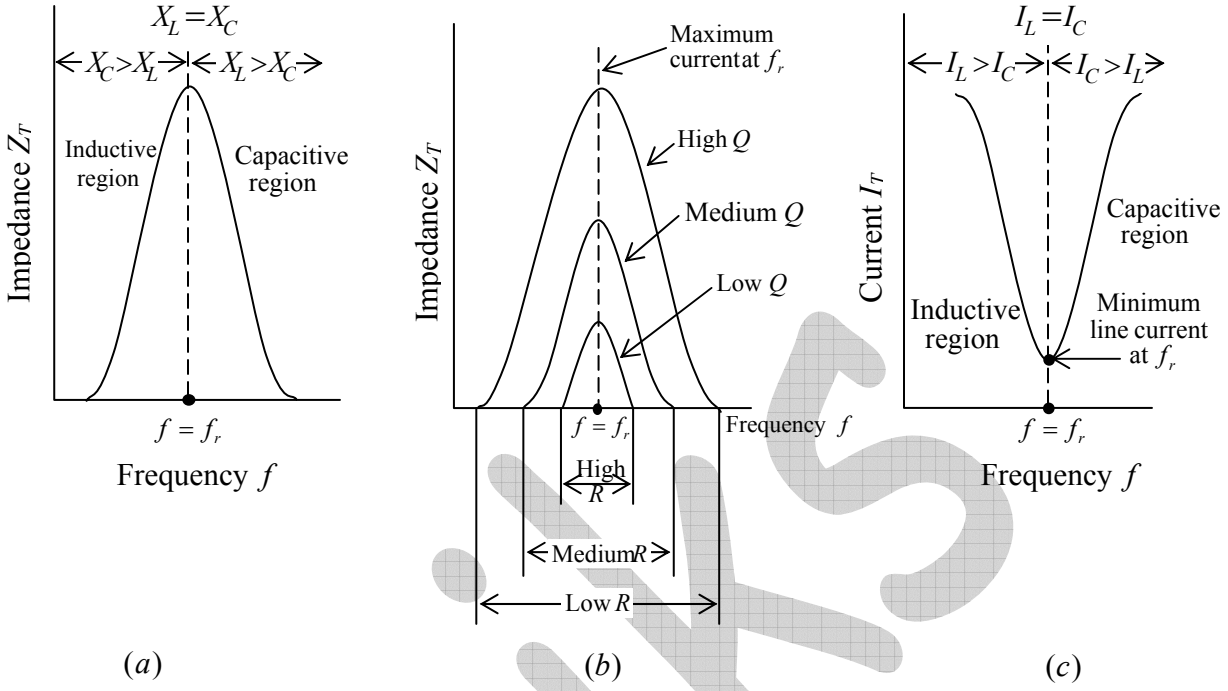


Figure: Impedance and current response curves of practical parallel LC circuit at resonance

9.6.2.3 Q of Parallel Circuit

For a parallel resonant circuit in which R is very low compared with X_L,

$$Q = \frac{X_L}{R}$$

where R is the resistance of the coil in series with X_L. If the resistance of the source supply is very high and there is no other resistance branch shunting the tuned circuit, the Q of the parallel resonant circuit is the same as the Q of the coil.

9.6.3 Bandwidth and Power of Resonant Circuit

The width of the resonant band of frequencies centered around f_r is called the bandwidth of the tuned circuit. In figure *a* shown below, the group of frequencies with a response of 70.7 percent of maximum or more is considered the bandwidth of the tuned circuit. For a series resonant circuit, the bandwidth is measured between the two frequencies f_1 and f_2 producing 70.7 percent of the maximum current at f_r (figure *b* shown below). For a

parallel resonant circuit, the bandwidth is measured between the two frequencies, allowing 70.7 percent of the maximum total impedance at f_r (figure *c* shown below).

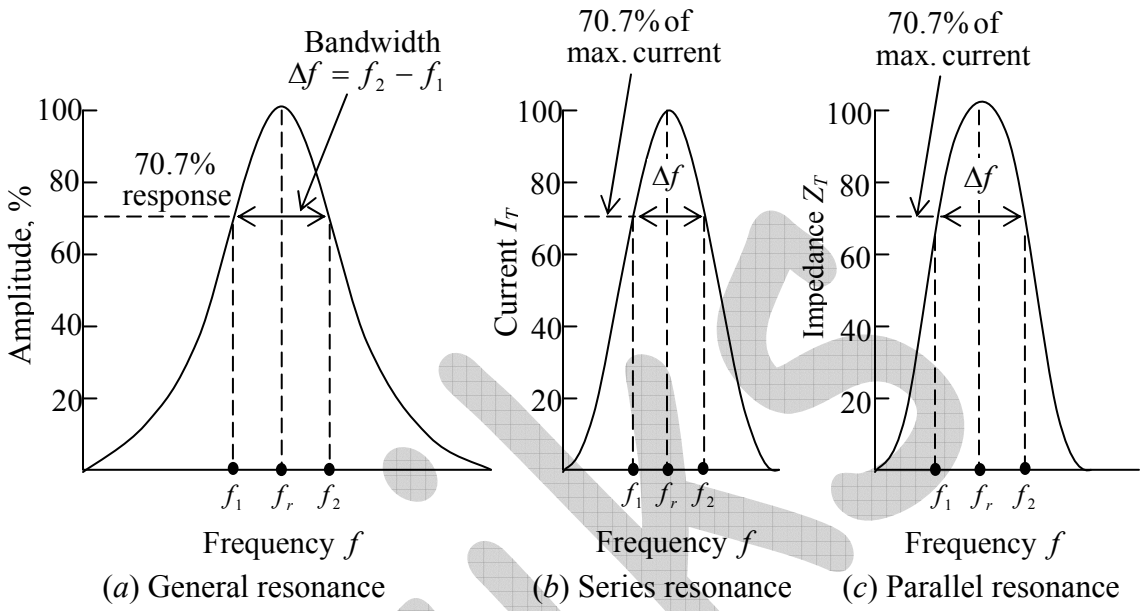


Figure: Bandwidth of a tuned *LC* circuit

At each frequency f_1 and f_2 the net capacitive or net inductive reactance equals the resistance. Then Z_T of the series *RLC* resonant circuit is $\sqrt{2}$ or 1.4 times greater than *R*. The current then is $I/\sqrt{2} = 0.707I$. Since power is I^2R or V^2/R and $(0.707)^2 = 0.50$, the bandwidth at 70.7 percent response in current or voltage is also the bandwidth of half-power points.

Bandwidth (*BW*) in terms of *Q* is

$$BW = f_2 - f_1 = \Delta f = \frac{f_r}{Q}$$

High *Q* means narrow bandwidth, whereas low *Q* yields greater bandwidth.

Either f_1 or f_2 is separated from f_r by one-half of the total bandwidth, so these edge frequencies can be calculated.

$$f_1 = f_r - \frac{\Delta f}{2} \quad \text{and} \quad f_2 = f_r + \frac{\Delta f}{2}$$

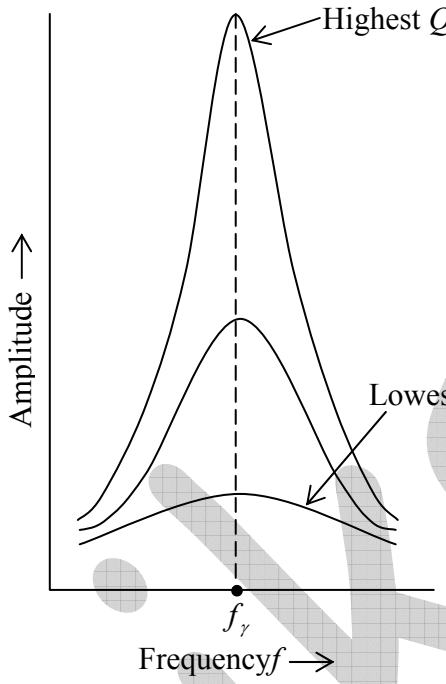


Figure: Resonant response curves: higher Q provides sharper resonance, lower Q provides broader response

Table: Comparison of Series and Parallel Resonance

Series Resonance	Parallel Resonance(Q>10)
$f_r = \frac{1}{2\pi\sqrt{LC}}$	$f_r = \frac{1}{2\pi\sqrt{LC}}$
I maximum at f_r with $\theta = 0^0$	I_T maximum at f_r with $\theta = 0^0$
Impedance Z minimum at f_r	Impedance Z_T maximum at f_r
$Q = \frac{X_L}{R}$	$Q = \frac{X_L}{R}$
Q rise in voltage= QV_T	Q rise in impedance= QX_L
Bandwidth $\Delta f = \frac{f_r}{Q}$	Bandwidth $\Delta f = \frac{f_r}{Q}$
Capacitive below f_r, but inductive above f_r	Inductive below f_r, but capacitive above f_r

MCQ (Multiple Choice Questions)

Q1. A voltage of 15V ac is applied to a $150\,\mu H$ coil connected in series with a $169\,pF$ capacitor. The total series resistance is $7.5\,\Omega$ which includes the coil winding resistance, the resistance of the connecting leads, and the leakage resistance of the capacitor. The circuit is resonant at 1000 kHz. Then the behavior of circuit at frequency of 1200 kHz is

(a) Resistive (b) Capacitive

(c) Inductive (d) both Inductive and Capacitive

Q2. A voltage of 15V ac is applied to a $150\,\mu H$ coil connected in series with a $169\,pF$ capacitor. The total series resistance is $7.5\,\Omega$ which includes the coil winding resistance, the resistance of the connecting leads, and the leakage resistance of the capacitor. The circuit is resonant at 1000 kHz. Then the behavior of circuit at frequency of 1000 kHz is

(a) Resistive (b) Capacitive

(c) Inductive (d) both Inductive and Capacitive

Q3. In a simple radio receiver tuning circuit (series RLC circuit), resistance is $7.5\,\Omega$, inductance is $400\,\mu H$ and the circuit is resonant at $500\,kHz$. Then the value of series capacitor is:

(a) 252 pF (b) 254 pF

(c) 256 pF (d) 258 pF

Q4. A voltage of $15\,V$ ac is applied to a $150\,\mu H$ coil connected in series with a 169 pF capacitor. The total series resistance is $7.5\,\Omega$. The circuit is resonant at $1000\,kHz$. The magnitude of current at 1000 kHz is

(a) 2 A (b) 4 A (c) 6 A (d) 8 A

Q5. An LCR series circuit with $C = 0.1\,\mu F$ and having a resonance frequency of $1.6\,kHz$ would have a quality factor (Q) of 10, if the series resistance is chosen to be

(a) 10 Ω (b) 50 Ω (c) 100 Ω (d) 160 Ω

Q6. A parallel resonant circuit appears as infinite impedance at the resonant frequency. A $400\ \mu H$ coil and a $25\ pF$ capacitor are placed in parallel to form a "wave trap" in an antenna. The resistance of the circuit is negligible. Then the resonant frequency is

(a) 1550 kHz (b) 1570 kHz

(c) 1590 kHz (d) 1610 kHz

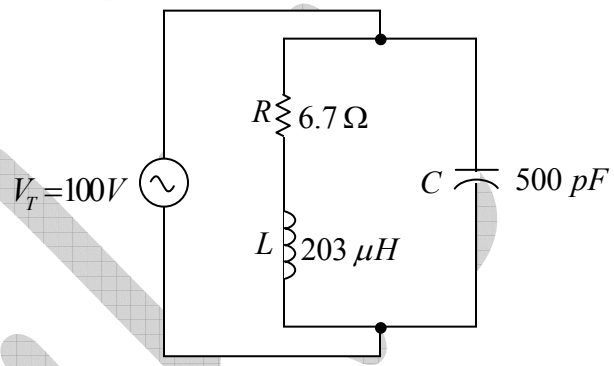
Q7. For the circuit shown in figure the resonant frequency is:

(a) 300 kHz

(b) 350 kHz

(c) 450 kHz

(d) 500 kHz



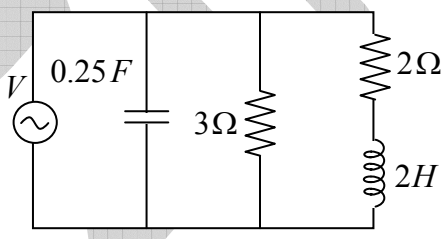
Q8. Find the resonance frequency (in rad/sec) of the circuit shown in the figure below

(a) 1.0

(b) 1.6

(c) 2.0

(d) 2.6



Q9. An LC circuit resonant at $1000\ kHz$ has a Q of 100. Then the edge frequencies are respectively

(a) 995 kHz, 1005 kHz (b) 990 kHz, 1010 kHz

(c) 995 kHz, 1010 kHz (d) 990 kHz, 1005 kHz

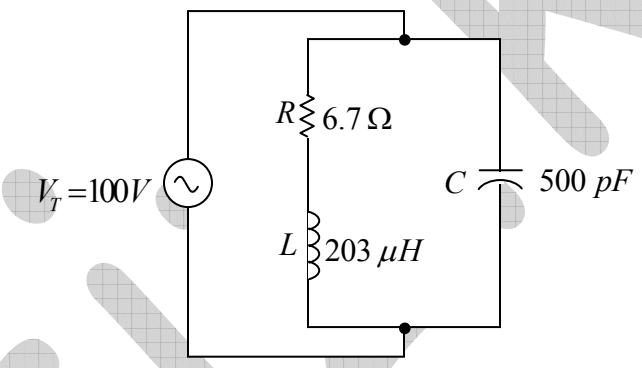
NAT (Numerical Answer Type)

Q10. A 150 Ω resistor, a 10 μF capacitor and a 0.1 H inductor are connected in series to an a.c. source operating at an angular frequency ω. Then the value of ω (in rad/s) for which the combination acts as a pure resistive load is……

Q11. In a simple radio receiver tuning circuit (series RLC circuit), resistance is $7.5\,\Omega$, inductance is $200\,\mu H$ and the capacitor is set at $200\,pF$. Then the resonant frequency is: ……..kHz

Q12. The capacitance of a parallel resonant circuit used as a wave trap in an antenna circuit is $400\,pF$. Then the value of parallel inductance in order to reject a $800\,kHz$ wave is…… μH

Q13. For the circuit shown in figure the impedance of the circuit is……$k\Omega$



Q14. A coil with a Q of 71.6 is connected in parallel with a capacitor to produce resonance at $356\,kHz$. The impedance at resonance is found to be $64\,k\Omega$. Then the value of the capacitor is…… pF

MSQ (Multiple Select Questions)

Q15. A 150 Ω resistor, a 10 μF capacitor and a 0.1 H inductor are connected in series to an a.c. source operating at an angular frequency ω. The a.c. source is operated at a peak voltage of $300\sqrt{2}\ V$ and a frequency equal to half the resonance frequency of the circuit. Then which of the following statements are true.

(a) The circuit is operating at $\omega = 5\times10^2\ rad/s$

(b) The peak value of current in the circuit is $2\ A$

(c) The phase difference between the current and voltage is -45^0

(d) The peak value of voltage across the inductor is $100\,V$

Q16. A 150 Ω resistor, a 10 μF capacitor and a 0.1 H inductor are connected in series to an a.c. source operating at $150\sqrt{2}\sin 500t$ volts. Then which of the following statements are true.

(a) The circuit is resonant at $\omega = 10^3\ rad/s$

(b) The circuit is resonant at $\omega = 5\times10^2\ rad/s$

(c) The peak value of current at operating frequency in the circuit is $2\ A$

(d) The peak value of current at operating frequency in the circuit is $1\,A$

Q17. A 150 Ω resistor, a 10 μF capacitor and a 0.1 H inductor are connected in series to an a.c. source operating at $150\sqrt{2}\sin 500t$ volts. Then which of the following statements are true.

(a) The phase difference between the current and voltage is -60^0

(b) The phase difference between the current and voltage is -45^0

(c) The peak value of voltage across the inductor is $100\,V$

(d) The peak value of voltage across the inductor is $50\,V$

Solutions

MCQ (Multiple Choice Questions)

Ans. 1: (c)

Ans. 2: (a)

Ans. 3: (b)

Solution: $f_r = \dfrac{1}{2\pi\sqrt{LC}} \Rightarrow C = \dfrac{0.0254}{f_r^2 L} = 254\ pF$

Ans. 4: (b)

Solution: At resonance $Z = R = 7.5\ \Omega$

$$\text{Current } I = \frac{V_T}{R} = \frac{15}{7.5} = 2\ A$$

Ans. 5: (c)

Solution: $Q = \dfrac{1}{2\pi fRC} \Rightarrow R = \dfrac{1}{2\pi fQC} = 100\Omega$

Ans. 6: (c)

Solution: $f_r = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{0.159}{\sqrt{LC}} = 1590\ kHz$

Ans. 7: (d)

Solution: $f_r = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}} = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{\left(203\times10^{-6}\right)\left(500\times10^{-12}\right)} - \left(\dfrac{6.7}{203\times10^{-6}}\right)^2} = 500\ kHz$

Ans. 8: (a)

Solution: $\omega_r = \sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}} = \sqrt{\dfrac{1}{2\times0.25} - \dfrac{4}{4}} = 1.0$

Ans. 9: (a)

Solution: $\Delta f = \dfrac{f_r}{Q} = \dfrac{1000\times10^3}{100} = 10\ kHz$

$$f_1 = f_r - \frac{\Delta f}{2} = 995\ kHz \text{ and } f_2 = f_r + \frac{\Delta f}{2} = 1005\ kHz$$

NAT (Numerical Answer Type)

Ans. 10: **1000**

Solution: At resonance $\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1\times 10\times 10^{-6}}} = 10^3 \ rad/s$

Ans. 11: **795**

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{0.159}{\sqrt{LC}} = 795 \ kHz$$

Ans. 12: **99.2**

Solution: $f_r = \frac{1}{2\pi\sqrt{LC}} \Rightarrow L = \frac{0.0254}{f_r^2 C} = 99.2 \ \mu H$

Ans. 13: **60.6**

Solution: $Z_T = \frac{L}{RC} = \frac{203\times 10^{-6}}{6.7\left(500\times 10^{-12}\right)} = 60.6 \ k\Omega$

Ans. 14: **500**

Solution: $Z_T = \frac{Q}{2\pi f_r C} \Rightarrow C = \frac{Q}{2\pi f_r Z_T} = \frac{71.6}{6.28\left(356\times 10^3\right)\left(64\times 10^3\right)} = 500 \ pF$

MSQ (Multiple Select Questions)

Ans. 15: (a), (b), (c), (d)

Solution: (a) At resonance $\omega' = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1\times 10\times 10^{-6}}} = 10^3 \ rad/s \ \ \omega = \frac{\omega'}{2} = 5\times 10^2 \ rad/s$.

(b) Given that peak voltage $= 300\sqrt{2} \ V$ and frequency $\omega = 5\times 10^2 \ rad/s$.

$$R = 150\,\Omega, \ X_C = \frac{1}{\omega C} = \frac{1}{5\times 10^2 \times 10^{-5}} = 200\,\Omega, \ X_L = \omega L = 5\times 10^2 \times 0.1 = 50\,\Omega$$

$$Z = \sqrt{R^2 + \left(X_C - X_L\right)^2} = \sqrt{150^2 + \left(200-50\right)^2} = \sqrt{2\times 150^2} = 150\sqrt{2} \ \Omega.$$

Peak current $I_M = \frac{300\sqrt{2}}{150\sqrt{2}} = 2 \ A$

(c) $\theta = \arctan\left(-\frac{X_C - X_L}{R}\right) = \arctan\left(-\frac{200-50}{150}\right) = -45^0$.

(d) Peak voltage across the inductor $= I_M X_L = 2\times 50 = 100\ V$

Ans. 16: (a), (d)

Solution: At resonance $\omega' = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{0.1\times 10\times 10^{-6}}} = 10^3\ rad/s$

Given that peak voltage $= 150\sqrt{2}\ V$ and operation frequency $\omega = 5\times 10^2\ rad/s$

$R = 150\,\Omega,\ X_C = \frac{1}{\omega C} = \frac{1}{5\times 10^2\times 10^{-5}} = 200\,\Omega,\ X_L = \omega L = 5\times 10^2\times 0.1 = 50\,\Omega$

$Z = \sqrt{R^2 + (X_C - X_L)^2} = \sqrt{150^2 + (200-50)^2} = \sqrt{2\times 150^2} = 150\sqrt{2}\ \Omega$.

Peak current $I_M = \frac{150\sqrt{2}}{150\sqrt{2}} = 1\ A$

Ans. 17: (b), (d)

Solution: Given that peak voltage $= 150\sqrt{2}\ V$ and frequency $\omega = 5\times 10^2\ rad/s$

$R = 150\,\Omega,\ X_C = \frac{1}{\omega C} = \frac{1}{5\times 10^2\times 10^{-5}} = 200\,\Omega,\ X_L = \omega L = 5\times 10^2\times 0.1 = 50\,\Omega$

$Z = \sqrt{R^2 + (X_C - X_L)^2} = \sqrt{150^2 + (200-50)^2} = \sqrt{2\times 150^2} = 150\sqrt{2}\ \Omega$

Peak current $I_M = \frac{150\sqrt{2}}{150\sqrt{2}} = 1\ A$

Peak voltage across the inductor $= I_M X_L = 1\times 50 = 50\ V$

$\theta = \arctan\left(-\frac{X_C - X_L}{R}\right) = \arctan\left(-\frac{200-50}{150}\right) = -45^0$.

Chapter - 10

Simple DC circuits with R, L and C components

10. Circuit Analysis by Classical Method

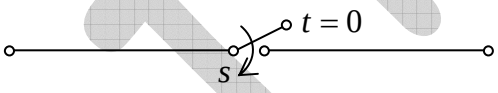
10.1 Introduction

Whenever a circuit is switched from one condition to another, either by a change in the applied source or a change in the circuit elements, there is a transition period during which the branch currents and element voltages change from their former values to new ones. This period is called the *transient*. After the transient is passed, the circuit is said to be in the *steady state*.

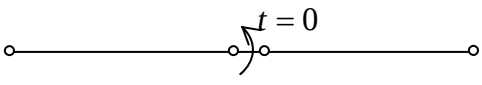
Now, the linear differential equation that describes the circuit will have two parts to its solution. The complementary function corresponds to the transient state and the particular solution corresponds to the steady state.

The *v-i* relation for an inductor or capacitor is a differential equation. A Circuit containing an inductor L or a capacitor C and resistor R will have current and voltage variables given by differential equations of the same form. It is a linear first order differential equation with constant coefficients when values of R, L and C are constant. L and C are storage element.. Circuits have two storage elements like one L and one C are referred to as second order circuits.

The circuit changes are assumed to occur at time $t = 0$ and represented by a switch.



Closed / ON



Open / OFF

$t = 0^-$; the instant prior to $t = 0$ and

$t = 0^+$; the instant immediately after switching

Switching on or off an element or source in a circuit at $t = 0$ will not disturb the storage element so that $i_L\left(0^-\right) = i_L\left(0^+\right)$ and $v_C\left(0^-\right) = v_C\left(0^+\right)$

Circuit element: Resistance

Resistance $R = \frac{V}{I}$

Power absorbed by the resistor $P = I^2R = \frac{V^2}{R}$

and energy lost in the resistance in form of heat is $W = \int_0^t Pdt = Pt = I^2Rt = \frac{V^2t}{R}$

Circuit element: Inductance

Voltage across inductance $v_L = -L\frac{di}{dt}$

In a pure inductive circuit with applied voltage v, $v + v_L = 0 \Rightarrow v = -v_L = L\frac{di}{dt}$

Power absorbed by inductor $P = vi = Li\frac{di}{dt} \Rightarrow W = \int_0^t Pdt = \int_0^t Li\frac{di}{dt} = \frac{1}{2}Li^2$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} v_L dt = \frac{1}{L}\int_0^t v_L dt + i_L(0^-)$$

Circuit element: Capacitance

$$C = \frac{q}{v},\ i = C\frac{dv}{dt} \Rightarrow v_C = \frac{1}{C}\int_0^t i_C dt + v_C(0^-)$$

$$P = vi = vC\frac{dv}{dt} \quad \text{and} \quad W = \frac{1}{2}Cv^2$$

10.2 Differential Equations

10.2.1 Type I-(First order Homogeneous Differential Equation)

$\frac{dy}{dt}(t)+Py(t)=0$ where P is any constant

$\frac{dy(t)}{y(t)}=-Pdt \Rightarrow \ln y(t)=-Pt+k'$, take $k'=\ln k \Rightarrow \ln y(t)=-Pt+\ln k$

$\Rightarrow y(t)=ke^{-Pt}$ where k is a constant

10.2.2 Type II-(First order Non Homogeneous Differential Equation)

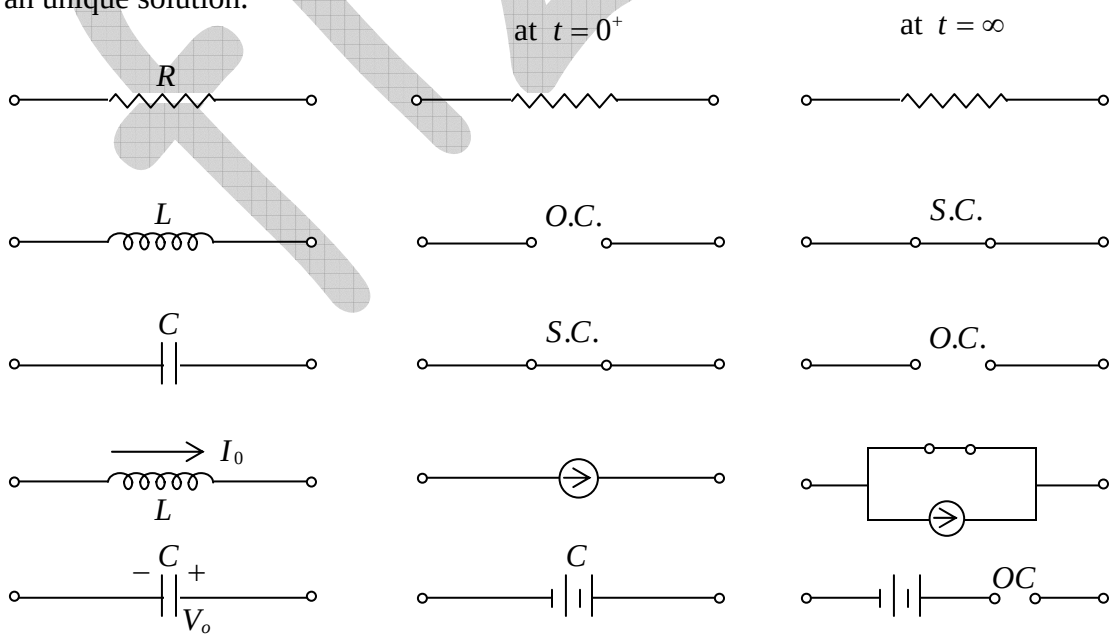
$\frac{dy}{dt}(t)+Py(t)=Q$ where P is a constant and Q may be a function of independent variable t or a constant.

$$\frac{d}{dt}\left[y(t)e^{Pt}\right]=Qe^{Pt} \Rightarrow y(t)e^{Pt}=\int Qe^{Pt}dt+k \Rightarrow y(t)=e^{-Pt}\underbrace{\int Qe^{Pt}dt}_{P.I.}+\underbrace{ke^{-Pt}}_{C.F.}$$

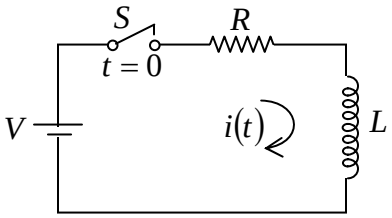
If Q is constant, then $y(t)=\frac{Q}{P}+ke^{-Pt}$

10.3 Initial Conditions in Circuits

Number of initial conditions required is equal to the order of the differential equation for an unique solution.



10.4 Transient Response of Series *R-L* Circuit having DC Excitation



If switch S is closed at $t=0$. Then the current through the circuit is $i(t)$. Applying KVL around the loop, we will get

$$-V+iR+L\frac{di}{dt}=0 \Rightarrow \frac{di}{dt}+\frac{R}{L}i=\frac{V}{L}\Rightarrow i(t)=\frac{V}{R}+ke^{-\frac{R}{L}t}$$

Now L behaves as open circuit (O.C.) at switching, $i(0^+)=0\Rightarrow k=-\frac{V}{R}$

Thus $i(t)=\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right)$

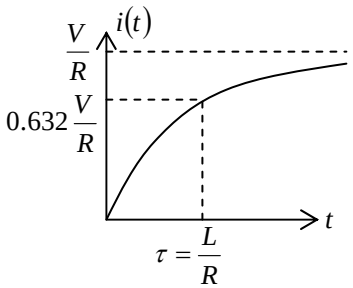
$V_R(t)=iR=V\left(1-e^{-\frac{R}{L}t}\right)$ and $V_L(t)=L\frac{di}{dt}=Ve^{-\frac{R}{L}t}$

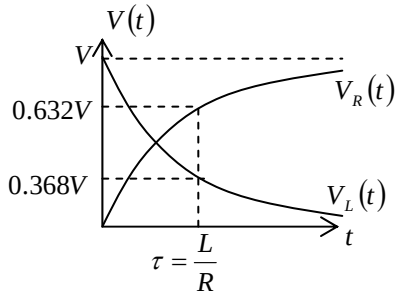
At $t=0$, $i(t)=0$, $V_L(t)=V$, $V_R(t)=0$

At $t=\infty$, $i(t)=\frac{V}{R}$ *and* $V_L(t)=0, V_R(t)=V$

When $t=\frac{L}{R}=\tau$ (time constant)

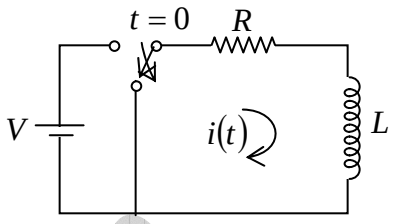
$i(t)=\frac{V}{R}\left(1-\frac{1}{e}\right)=0.632\frac{V}{R}$, $V_L(t)=\frac{V}{e}=0.368V$ *and* $V_R(t)=0.632V$





New Transient Condition

As circuit reaches at steady state (at $t=\infty$), suddenly switch is open and new condition at $t=0$ is shown.



Now $L\frac{di}{dt}+Ri=0 \Rightarrow i(t)=k'e^{-\frac{R}{L}t}$

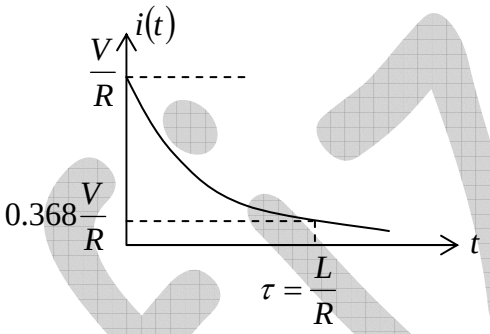
at $t=0^+$, the inductor keep the steady state

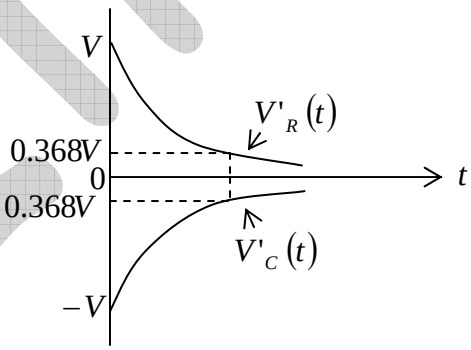
$$i(0^+)=i(0^-)=\frac{V}{R}$$

Thus $k'=\frac{V}{R} \Rightarrow \quad i(t)=\frac{V}{R}e^{-\frac{R}{L}t}$

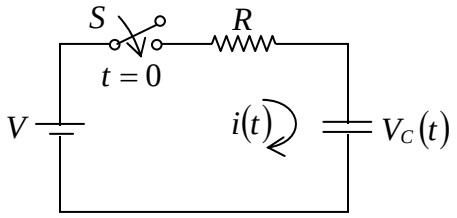
Corresponding voltages across the resistor and inductor are

$$V'_R(t)=i(t)R=Ve^{-\frac{R}{L}t} \quad \text{and} \quad V'_L(t)=L\frac{di}{dt}=-Ve^{-\frac{R}{L}t}$$





10.5 Transient Response of Series *R-C* Circuit having DC Excitation



If switch *S* is closed at $t=0$. Then the current through the circuit is $i(t)$. Applying KVL around the loop, we will get

$$-V+Ri(t)+\frac{1}{C}\int_{-\infty}^{t} i(t)dt=0 \text{ or } -V+Ri(t)+\frac{1}{C}\int_{0}^{t} i(t)dt+V_C(0^-)=0$$

Initially capacitor was uncharged $V_C(0^-)=0$

Differentiating above equation, we will get $R\frac{di}{dt}+\frac{1}{C}i(t)=0 \Rightarrow i(t)=ke^{-\frac{1}{RC}t}$

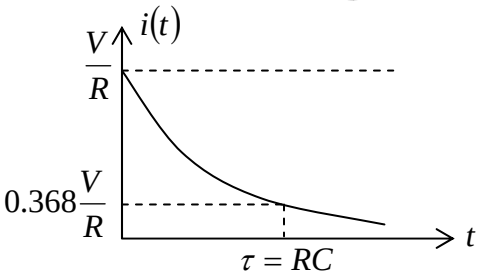
At $t=0^+,\ i(0^+)=\frac{V}{R}=k \Rightarrow i(t)=\frac{V}{R}e^{-\frac{1}{RC}t}$

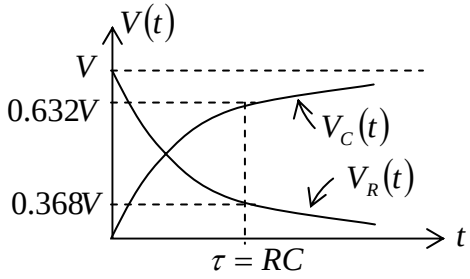
$$V_R(t)=i(t)R=Ve^{-\frac{1}{R}t} \text{ and } V_C(t)=\frac{1}{C}\int_0^t i(t)dt=V\left(1-e^{-\frac{1}{RC}t}\right)$$

At $t=0;\ i(t)=\frac{V}{R}$ and $V_C(t)=0,\quad V_R(t)=V$

At $t=\infty;\ i(t)=0$ and $V_C(t)=V,\quad V_R(t)=0$

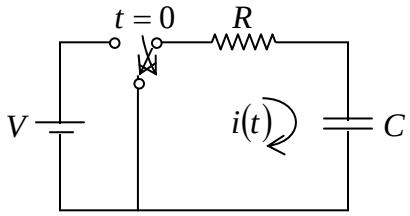
At $t=RC=\tau;\ i(t)=\frac{V}{R}e^{-1}=0.368\frac{V}{R}$ and $V_C(t)=0.632V,\ V_R(t)=0.368V$





New Transient Condition

As circuit reaches at steady state (at $t=\infty$), suddenly switch is open and new condition $t=0$ is shown.



Now $Ri(t)+\frac{1}{C}\int_0^t i(t)dt+V_C(0^+)=0$

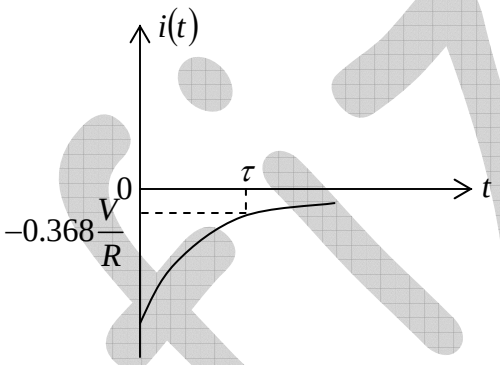
$\Rightarrow R\frac{di}{dt}+\frac{1}{C}i=0 \Rightarrow \quad i(t)=k'e^{-\frac{1}{RC}t}$

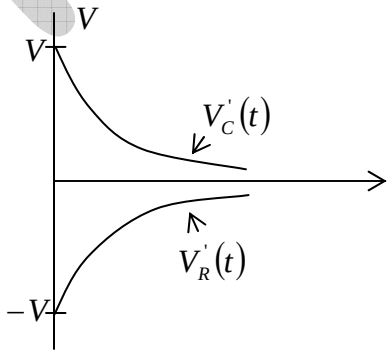
At $\quad t=0^+,\ V_C(0^+)=V$

$\Rightarrow \quad i(0^+)=i(0^-)=-\frac{V}{R}$ (Discharge is negative)

$\Rightarrow i(t)=-\frac{V}{R}e^{-\frac{1}{RC}t}$

$V'_R(t)=i(t)R=-Ve^{-\frac{1}{RC}t}$ and $V'_C(t)=\frac{1}{C}\int i(t)dt=Ve^{-\frac{1}{RC}t}$





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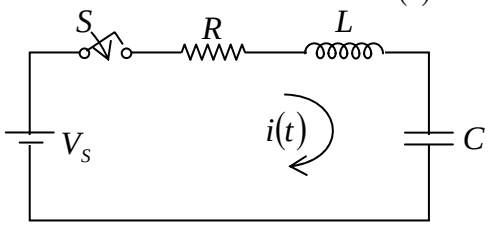
10.6 Transient Response of Series *RLC* Circuit having DC Excitation

Consider series RLC circuit $V_S = 2\,V$, $L = 2\,H$, $R = 6\,\Omega$ and $C = 0.25\,F$. At $t = 0$ switch S is closed then in the transient state find current $i(t)$.



Applying KVL, $V_S = L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt$ ..... (1)

$\Rightarrow \quad 2\frac{di}{dt} + 6i + \frac{1}{0.25}\int i\,dt = 2 \Rightarrow \quad 2\frac{d^2i}{dt^2} + 6\frac{di}{dt} + 4i = 0$

$\Rightarrow \quad \frac{d^2i}{dt^2} + 3\frac{di}{dt} + 2i = 0$ ..... (2)

$\Rightarrow \quad p^2 + 3p + 2 = 0 \quad \Rightarrow \quad p_1 = -1, p_2 = -2$

$\Rightarrow \quad i(t) = k_1e^{-t} + k_2e^{-2t}$ ..... (3)

At $t = 0$, $i(0^+) = 0$ (inductor is open circuited)

$\Rightarrow \quad k_1 + k_2 = 0$

From (1) $\Rightarrow \quad \frac{di}{dt}(0^+) = \frac{V_S}{L} = \frac{2}{2} = 1 \qquad \because Ri(0^+) = 0$ and $\frac{1}{C}\int_{-\infty}^{0^+} dt = 0$

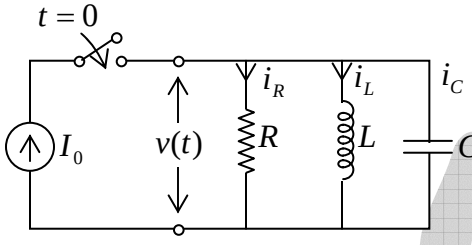
$\Rightarrow \quad \frac{di}{dt}(0^+) = 1 \Rightarrow -k_1 - 2k_2 = 1$

Thus, $k_1 = 1$, $k_2 = -1 \Rightarrow i(t) = e^{-t} - e^{-2t}$

and from (2) $\frac{d^2i}{dt^2}(0^+) + 3\times1 + 2\times0 = 0 \Rightarrow \frac{d^2i}{dt^2}(0^+) = -3$

10.7 Transient Response of Parallel *RLC* Circuit having DC Excitation

Consider Parallel *RLC* circuit $I_0 = 2\,A, L = \frac{1}{16}H$, $R = \frac{1}{16}\Omega$ and $C = 4\,F$. At $t = 0$ switch is closed then in the transient state find voltage $v(t)$.



Applying KCL, $i_R + i_L + i_C = I_0 \Rightarrow \frac{v(t)}{R} + \frac{1}{L}\int v\,dt + C\frac{dv}{dt} = I_0$ ..... (1)

$$\Rightarrow \quad \frac{d^2v}{dt} + 4\frac{dv}{dt} + 4v = 0 \text{ ..... (2)}$$

$$\Rightarrow \quad v(t) = k_1 e^{-2t} + k_2 t e^{-2t} \text{ ..... (3)}$$

$v(0^+) = 0$, Capacitor is S.C.

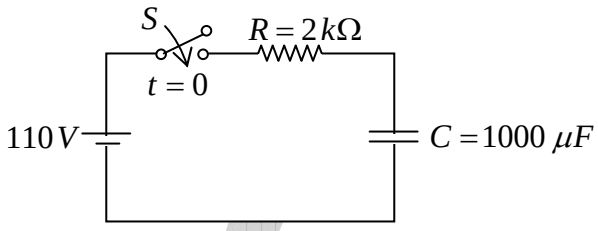
$$\Rightarrow \quad \frac{dv}{dt}(0^+) = \frac{I_0}{C} = \frac{2}{4} = \frac{1}{2} \quad \& \quad \frac{d^2v}{dt^2}(0^+) = -2$$

$$\Rightarrow \quad k_1 = 0 \text{ and } -2k_1 + k_2 = \frac{1}{2} \Rightarrow k_2 = \frac{1}{2}$$

$$\Rightarrow \quad v(t) = \frac{1}{2}te^{-2t}$$

MCQ (Multiple Choice Question)

Q1. The RC series circuit is shown in figure below. The capacitor is initially uncharged. At time $t = 0$, the switch S is closed. The voltage across resistor 1 sec after the circuit is closed is:



(a) $60.8\,V$

(b) $62.8\,V$

(c) $64.8\,V$

(d) $66.8\,V$

Q2. The RC series circuit shown in figure below has reached steady state when the capacitor has charged to 110 V, the switch is opened. The voltage across resistance and capacitance 1 sec after the circuit is opened are respectively:

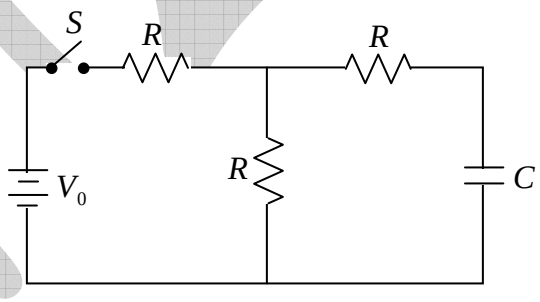
(a) $-66.8\,V$ and $+66.8\,V$

(b) $+66.8\,V$ and $-68.8\,V$

(c) $-60.8\,V$ and $+60.8\,V$

(d) $+60.8\,V$ and $-60.8\,V$

Q3. A capacitor C is connected to a battery V_0 through three equal resistors R and a switch S as shown below:



The capacitor is initially uncharged. At time $t = 0$, the switch S is closed. The voltage across capacitor as a function of time t for $t > 0$ is given by

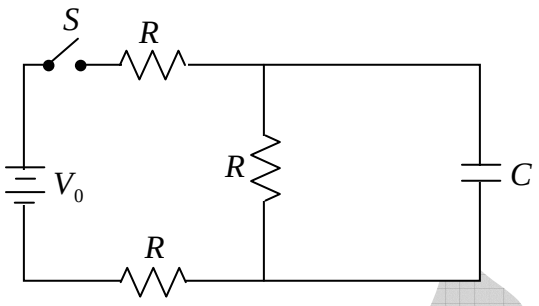
(a) $(V_0/2)\left[1-\exp(-3t/2RC)\right]$

(b) $(V_0/2)\left[1-\exp(-2t/3RC)\right]$

(c) $(V_0/2)\exp(-3t/2RC)$

(d) $(V_0/2)\exp(-2t/3RC)$

Q4. A capacitor C is connected to a battery V_0 through three equal resistors R and a switch S as shown below:

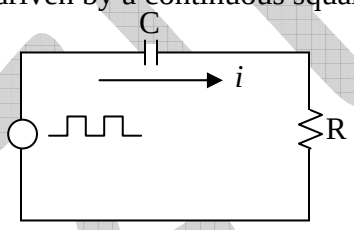


The capacitor is initially uncharged. At time $t = 0$, the switch S is closed. The voltage across the capacitor as a function of time t for $t > 0$ is given by

(a) $(V_0/2)\left[1-\exp(-3t/2RC)\right]$ (b) $(V_0/2)\left[1-\exp(-2t/3RC)\right]$

(c) $(V_0/3)\left[1-\exp(-3t/2RC)\right]$ (d) $(V_0/3)\left[1-\exp(-2t/3RC)\right]$

Q5. The RC circuit shown below is driven by a continuous square wave.



Which of the following figures is a correct representation of the current i in the circuit?

Figure 1

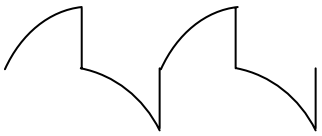


Figure 2

Figure 3

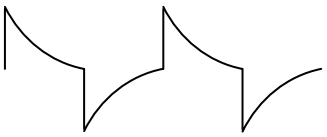
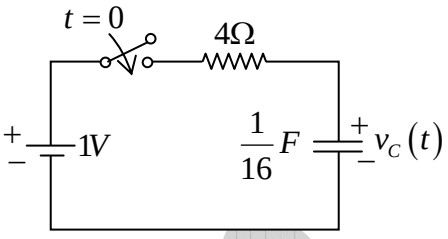


Figure 4

(a) Figure 1 (b) Figure 2 (c) Figure 3 (d) Figure 4

Q6. The RC series circuit is shown in figure below. The capacitor is initially charged to $9V$. At time $t=0$, the switch S is closed. Then in the transient state voltage across capacitor $v_C(t)$ is given by



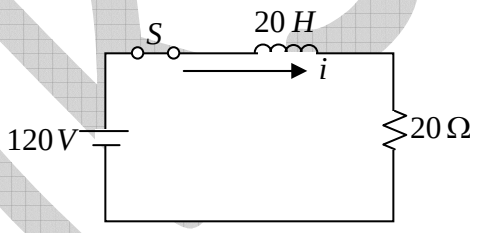
(a) $-2e^{-4t}$

(b) $2e^{-4t}$

(c) $1-8e^{-4t}$

(d) $1+8e^{-4t}$

Q7. The RL circuit shown below is driven by a dc source of voltage $120V$. Switch S is closed at $t=0$. The current through the circuit 2 sec after the circuit is closed is:



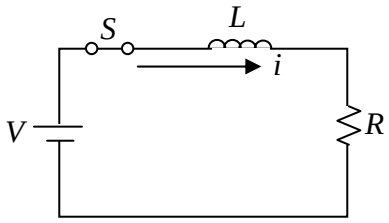
(a) $5.2\ A$

(b) $5.0\ A$

(c) $4.8\ A$

(d) $4.6\ A$

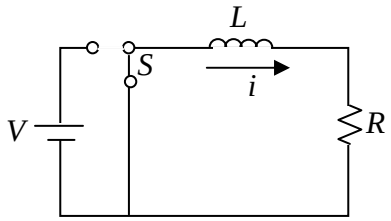
Q8. The RL circuit shown below is driven by a dc source of voltage V. Switch S is closed at $t=0$, then in the transient state current $i(t)$ is given by



(a) $\frac{V}{R}e^{-(R/L)t}$ (b) $\frac{V}{R}\left[1-e^{-(R/L)t}\right]$

(c) $\frac{V}{R}e^{-(L/R)t}$ (d) $\frac{V}{R}\left[1-e^{-(L/R)t}\right]$

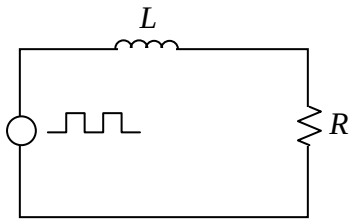
Q9. Initially RL circuit is driven by a dc source of voltage V. As circuit reaches steady state suddenly switch is opened and new condition at $t=0$ is shown in figure below. Then in the transient state voltage across resistor is given by



(a) $Ve^{-(L/R)t}$ (b) $V\left[1-e^{-(L/R)t}\right]$

(c) $Ve^{-(R/L)t}$ (d) $V\left[1-e^{-(R/L)t}\right]$

Q10. The *RL* circuit shown below is driven by a continuous square wave.



Which of the following figures is a correct representation of the output wave-form across the resistor?

Figure 1

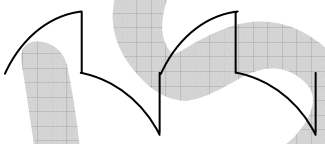


Figure 2

Figure 3

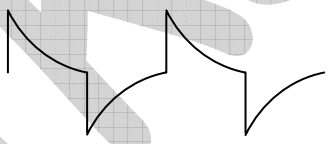
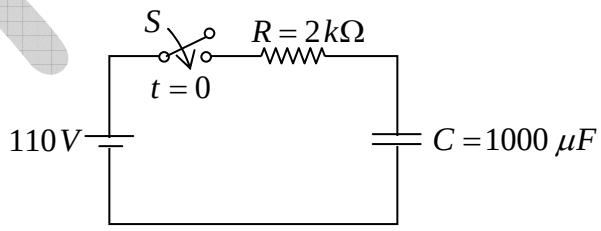


Figure 4

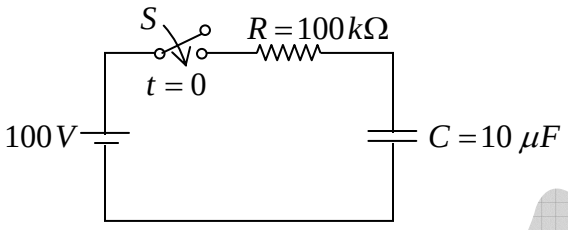
(a) Figure 1 (b) Figure 2 (c) Figure 3 (d) Figure 4

NAT (Numerical Answer Type)

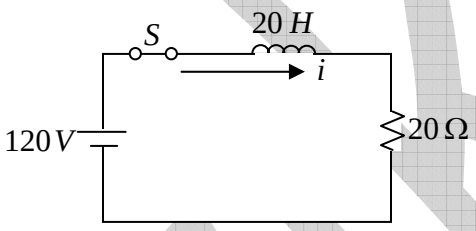
Q11. The *RC* series circuit is shown in figure below. The capacitor is initially uncharged. At time $t = 0$, the switch *S* is closed. The current through the circuit 1 sec after the circuit is closed is…….. *mA*



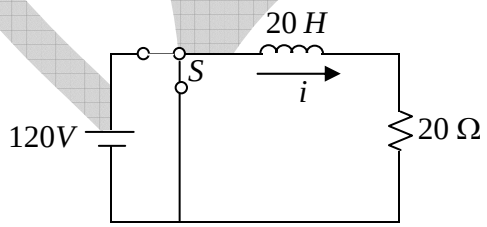
Q12. The RC series circuit is shown in figure below. The capacitor is initially uncharged. At time $t = 0$, the switch S is closed. How long…….. sec will it take for v_R to drop from 100 to 50 V ?



Q13. The RL circuit shown below is driven by a dc source of voltage 120 V . Switch S is closed at $t = 0$. The voltage across resistor 1 sec after the circuit is closed is……. Volts.



Q14. Initially RL circuit is driven by a dc source of voltage 120 V . As circuit reaches steady state suddenly switch is opened and new condition at $t = 0$ is shown in figure below. Then current in the circuit 2 sec after the switch is opened is…… Amp.



Solutions

MCQ (Multiple Choice Question)

Ans: (d)

Solution: $v_R = V\exp\left(-t/RC\right) = 110e^{\left(-\frac{1}{2000\times10^{-3}}\right)} = 110e^{(-0.5)} = 110\left(0.607\right) = 66.8\,V$

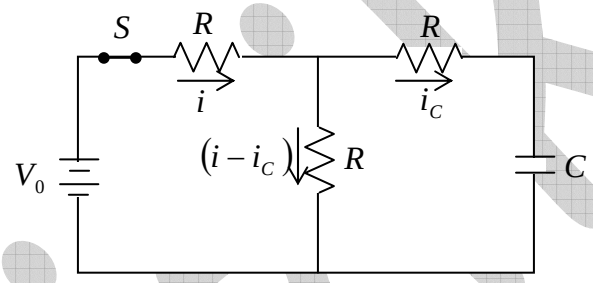
Ans. 2: (a)

Solution: $v_R = -V\exp\left(-t/RC\right) = -110e^{\left(-\frac{1}{2000\times10^{-3}}\right)} = -110e^{(-0.5)} = -110\left(0.607\right) = -66.8\,V$

$$v_C = V\exp\left(-t/RC\right) = 110e^{\left(-\frac{1}{2000\times10^{-3}}\right)} = 110e^{(-0.5)} = 110\left(0.607\right) = 66.8\,V$$

Ans. 3: (b)

Solution:



Applying KVL in loop 1, $\quad -V_0 + iR + (i-i_C)R = 0 \Rightarrow i = \dfrac{V_0 + i_C R}{2R}$

Applying KVL in loop 2, $\quad i_C R + \dfrac{1}{C}\int_0^t i_C\,dt - (i-i_C)R = 0$

$$\Rightarrow \frac{3}{2}i_C R + \frac{1}{C}\int_0^t i_C\,dt - \frac{V_0}{2} = 0 \Rightarrow \frac{3R}{2}\frac{di_C}{dt} + \frac{1}{C}i_C = 0$$

$$\Rightarrow \frac{di_C}{dt} + \frac{2}{3RC}i_C = 0 \Rightarrow i_C = K\exp\left(-\frac{2}{3RC}t\right)$$

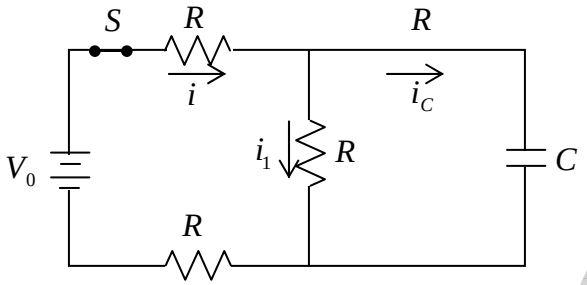
Since at $t=0,\ i(0) = \dfrac{2V_0}{3R}$ *and* $i_C(0) = \dfrac{V_0}{3R} \Rightarrow K = \dfrac{V_0}{3R} \Rightarrow i_C = (V_0/3R)\exp\left(-2t/3RC\right)$

Thus $v_C(t) = \dfrac{1}{C}\int_0^t (V_0/3R)\exp\left(-2t/3RC\right)dt = \dfrac{V_0}{3RC}\times -\dfrac{3RC}{2}\left[\exp\left(-2t/3RC\right)-1\right]$

$$\Rightarrow v_C(t) = (V_0/2)\left[1-\exp\left(-2t/3RC\right)\right]$$

Ans. 4: (c)

Solution:



Apply KCL, so $i = i_1 + i_C$

Applying KVL in loop 1, $-V_0 + (i_1 + i_C)R + i_1 R + (i_1 + i_C)R = 0$

$$\Rightarrow -V_0 + 3i_1 R + 2i_C R = 0 \Rightarrow -V_0 + \frac{3}{C}\int_0^t i_C dt + 2i_C R = 0 \qquad \left(\because i_1 R = \frac{1}{C}\int_0^t i_C dt\right)$$

$$\Rightarrow \frac{3}{C} i_C + +2R\frac{di_C}{dt} = 0 \Rightarrow \frac{di_C}{dt} + \frac{3}{2RC} i_C = 0 \Rightarrow i_C = K\exp\left(-\frac{3}{2RC}t\right)$$

Since at $t = 0$, $i_C(0) = \frac{V_0}{2R} \Rightarrow K = \frac{V_0}{2R} \Rightarrow i_C = (V_0/2R)\exp(-3t/2RC)$

$$\Rightarrow v_C = \frac{1}{C}\int_0^t i_C dt = (V_0/3)\left[1 - \exp(-3t/2RC)\right]$$

Ans. 5: (d)

Ans. 6: (d)

Solution: Applying KVL, $4i(t) + \underbrace{\frac{1}{C}\int_0^t i(t)\,dt}_{v_C(t)} + 9 = 1 \Rightarrow \frac{di}{dt} + 4i(t) = 0 \Rightarrow i(t) = ke^{-4t}$

Initial current $i(0^+) = \frac{1-9}{4} = -2 = k \Rightarrow i(t) = -2e^{-4t}$

Thus $v_C(t) = 9 + \frac{1}{C}\int_0^t i(t)\,dt = 1 + 8e^{-4t}$

Ans. 7: (a)

Solution: $i=\frac{V}{R}\left[1-e^{-\left(R/L\right)t}\right]=\frac{120}{20}\left[1-e^{-\left(20/20\right)2}\right]=6\left[1-0.135\right]=5.19\ A$

Ans. 8: (b)

Solution: Apply KVL, $-V+L\frac{di}{dt}+iR=0\Rightarrow\frac{di}{dt}+\frac{R}{L}i=\frac{V}{L}\Rightarrow i(t)=\frac{V}{R}+Ke^{-\left(R/L\right)t}$

At $t=0^+$ inductor behaves as an open circuit thus $i\left(0^+\right)=0\Rightarrow 0=\frac{V}{R}+K\Rightarrow K=-\frac{V}{R}$

$\Rightarrow i(t)=\frac{V}{R}\left[1-e^{-\left(R/L\right)t}\right]$

Ans. 9: (c)

Solution: Apply KVL, $L\frac{di}{dt}+iR=0\Rightarrow\frac{di}{dt}=-\frac{R}{L}i\Rightarrow i(t)=Ke^{-\left(R/L\right)t}$

At $t=0^+$ inductor behaves as an open circuit thus $i\left(0^+\right)=\frac{V}{R}\Rightarrow K=\frac{V}{R}$

$\Rightarrow i(t)=\frac{V}{R}e^{-\left(R/L\right)t}\Rightarrow v_R=i(t)R=Ve^{-\left(R/L\right)t}$

Ans. 10: (a)

NAT (Numerical Answer Type)

Ans. 11: 33.4

Solution: $i=\left(V/R\right)\exp\left(-t/RC\right)=\frac{110}{2000}e^{\left(-\frac{1}{2000\times10^{-3}}\right)}=\frac{110}{2000}e^{(-0.5)}=\frac{110}{2000}\left(0.607\right)=33.4\ mA$

Ans. 12: 0.693

Solution: $t=RC\ln\left(\frac{V}{v_R}\right)=10^5\times10^{-5}\ln\left(\frac{100}{50}\right)=\ln\left(2\right)=0.693\ \text{sec}$

Ans. 13: 75.8

Solution: $v_R\left(t\right)=i(t)R=V\left[1-e^{-\left(R/L\right)t}\right]=120\left[1-e^{-\left(20/20\right)1}\right]=120\left[1-0.368\right]=75.8V$

Ans. 14: 0.81

Solution: $i(t)=\frac{V}{R}e^{-\left(R/L\right)t}=\frac{120}{20}e^{-\left(20/20\right)2}=6\left(0.135\right)=0.81A$